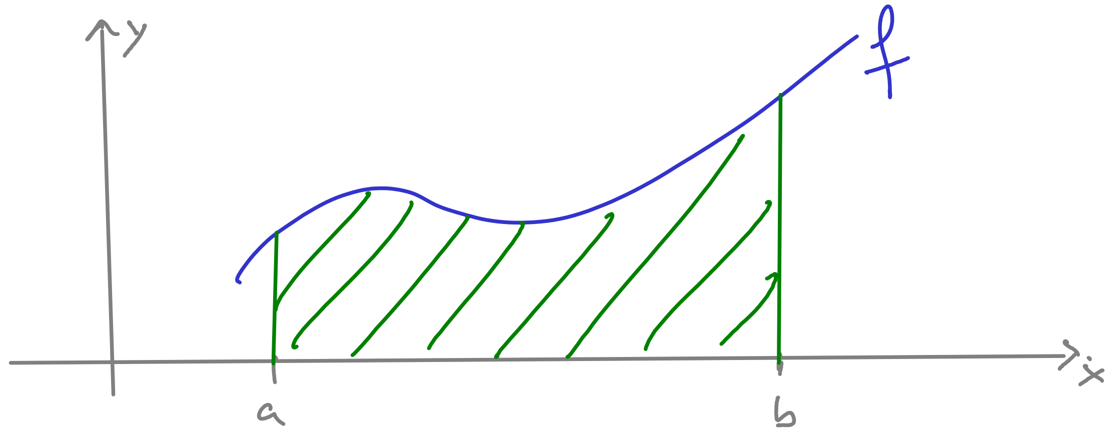
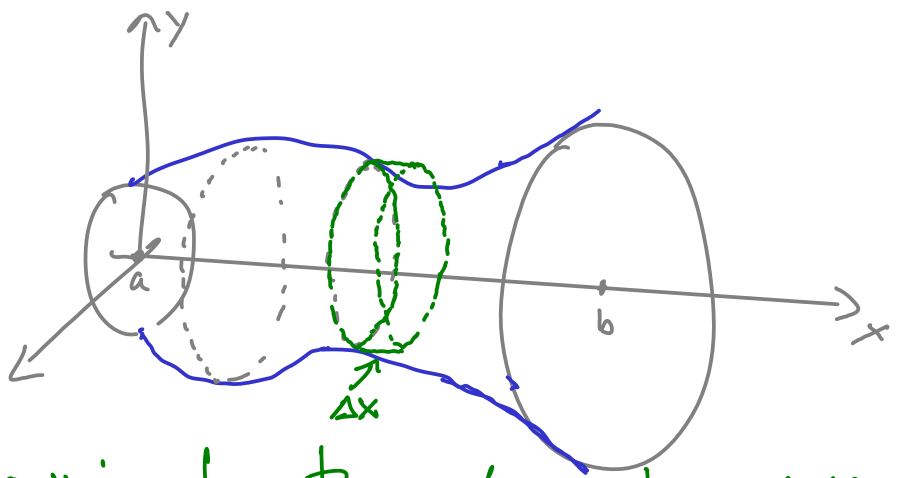
Volumes of revolution

Consider the following problem.

We want to know the volume of the solid obtained by rotating the area between the graph of a function f, the x-axis and the two lines $x=a$ and $x=b$ around the x-axis.





Idea: Approximate the volume by adding the volumes of thin discs.
Then take the limit when Δx, the thickness of the discs, shrinks to zero.

If we use n discs, then $\Delta x = \frac{b-a}{n}$, i.e.

$a = x_0$, $a + \Delta x = x_1$, ..., $x_i = a + i\Delta x$, ..., $b = x_n$

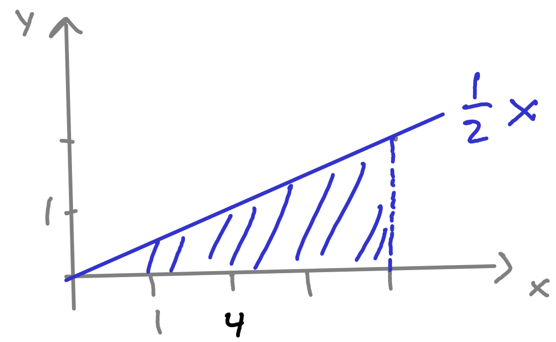
is our subdivision of the interval $[a,b]$, then the approximate volume is

$$V_n = \sum_{i=0}^{n-1} \pi \left[f(x_i)\right]^2 \Delta x .$$

When we let $n \to \infty$, then this becomes

$$V = \int_a^b \pi \left[f(x)\right]^2 dx .$$

Exaples: (1) A cone, say obtained by revolving the area underneath $f(x) = \frac{1}{2}x$ from 0 to 4.



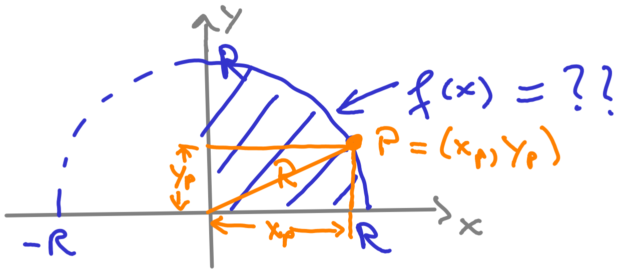
$$V = \pi \int_0^4 \left(\frac{1}{2}x\right)^2 dx = \pi \int_0^4 \frac{1}{4}x^2\, dx = \pi \left[\frac{1}{12}x^3\right]_0^4$$

$$= \frac{16}{3}\pi$$

Formula from tables:

$V_{cone} = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\, 2^2\, 4 = \frac{16}{3}\pi$

② Sphere (ball) of radius R.



$$R^2 = x_p^2 + y_p^2$$

Problem: Which function f should we take to get a quarter circle?

$$f(x) = y = \sqrt{R^2 - x^2}$$

So the volume of a ball of radius R is

$$V_{ball} = 2\pi \int_0^R \sqrt{R^2 - x^2}^{\,2} \, dx$$

$$= 2\pi \int_0^R (R^2 - x^2) \, dx$$

$$= 2\pi \left[R^2 x - \frac{1}{3} x^3 \right]_0^R$$

$$= 2\pi \left(R^3 - \frac{1}{3} R^3 - (0) \right)$$

$$= \frac{4}{3} \pi R^3$$