Trigonometric substitution

Yesterday we found

$$\int \cos(\omega t)\, dt = \frac{1}{\omega} \sin(\omega t)$$

This is really a substitution: Put $u = \omega t$, then $du = \omega\, dt$ or $dt = \frac{1}{\omega} du$, so that

$$\int \cos(\omega t)\, dt = \int \cos(u) \frac{1}{\omega} du = \frac{1}{\omega} \sin(u) = \frac{1}{\omega} \sin(\omega t)$$

Similarly, $\int \sin(\omega t + \varphi)\, dt = -\frac{1}{\omega} \cos(\omega t + \varphi) + c$

small phi, capital Φ

using the substitution $u = \omega t + \varphi$.

Problem: Find $\int \sqrt{a^2 - x^2}\, dx$, $0 < a = \text{const}$.

Sol^n: Put $x = a \sin(\alpha)$. Then $\frac{dx}{d\alpha} = a \cos(\alpha)$ and

$$\int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - a^2 \sin^2(\alpha)}\; a \cos(\alpha)\, d\alpha$$

$$= \int \sqrt{a^2(1 - \sin^2(\alpha))}\; a \cos(\alpha)\, d\alpha$$

$$= \int a^2 |\cos(\alpha)| \cos(\alpha)\, d\alpha$$

$$= \int a^2 \cos^2(\alpha)\, d\alpha \qquad \text{if } \cos(\alpha) \geq 0$$

$$= a^2 \int \frac{1}{2} (1 + \cos(2\alpha))\, d\alpha$$

$$= \frac{a^2}{2}\left(\alpha + \frac{1}{2}\sin(2\alpha)\right) + c$$

$$= \frac{a^2}{2}\left(\arcsin\left(\frac{x}{a}\right) + \frac{1}{2}\sin\left(2\arcsin\left(\frac{x}{a}\right)\right)\right) + c$$

From: $x = a\sin(\alpha)$ we get $\frac{x}{a} = \sin(\alpha)$

$$\arcsin\left(\frac{x}{a}\right) = \alpha$$

arcsin is the inverse function of sin, i.e.

$$\arcsin(\sin(x)) = x = \sin(\arcsin(x))$$

Some people write $\sin^{-1}(x)$ for $\arcsin(x)$

not to be confused with $(\sin(x))^{-1} = \frac{1}{\sin(x)} \neq \arcsin(x)$

Inverse functions:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "f"] & \mathbb{R} \arrow[r, "g"] & \mathbb{R} \\
x \arrow[r, mapsto] \arrow[rr, bend right, "g \circ f"'] & f(x) \arrow[r, mapsto] & g(f(x)) = (g \circ f)(x)
\end{tikzcd}$$

Eg. $f(x) = x^2 + 3$ and $g(t) = \sin(t)$

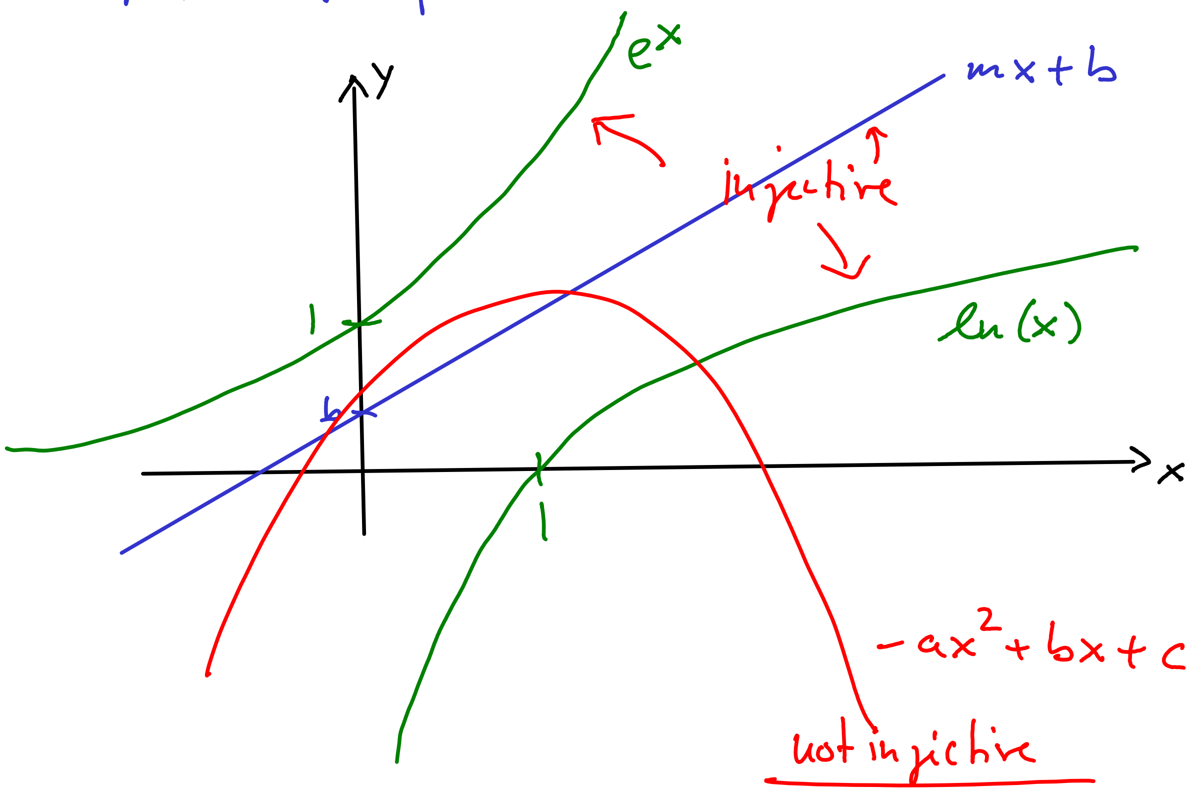
Then $(g \circ f)(x) = \sin(x^2 + 3)$

If $f(x_1) = f(x_2) = a$, then given only the information a, we cannot decide whether it was produced by f from x_1 or x_2.

<u>Def</u>: A function f is called <u>injective</u> if

$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2 .$$

This means that every horizontal line meets the graph of f at most once.



<u>Lines</u>: $y = mx+b$, then $x = \frac{y-b}{m} = \frac{1}{m}y - \frac{b}{m}$

So if $f(x) = mx+b$, then $g(x) = \frac{1}{m}x - \frac{b}{m}$ is the inverse function of f.

$$(g \circ f)(x) = g(mx+b) = \frac{mx+b}{m} - \frac{b}{m} = x$$

Logs & Exponentials:

$\ln(x)$ is the inverse function of $\exp(x) = e^x$

$\ln(e^x) = x$ and $e^{\ln x} = x$

arcsin: $\arcsin(x)$ is only defined for $-1 \leq x \leq 1$

and produces only values $-\frac{\pi}{2}$ and $\frac{\pi}{2}$