Some trigonometric integrals

Problem: Find $\int \cos^3(x) \sin^4(x)\, dx$.

Observation: $\frac{d}{dx}\left(\sin^k(x)\right) = k \sin^{k-1}(x) \cos(x)$

Similarly $\frac{d}{dx}\left(\cos^k(x)\right) = -k \cos^{k-1}(x) \sin(x)$

$\cos^k(x) = \left(\cos(x)\right)^k$
$k \geq 1$

Sol^n: $\int \cos^3(x) \sin^4(x)\, dx$

$$= \int \cos^2(x) \cos(x) \sin^4(x)\, dx$$

$$= \int \left(1 - \sin^2(x)\right) \cos(x) \sin^4(x)\, dx$$

use $\cos^2(x) = 1 - \sin^2(x)$

$$= \int \left(\cos(x) \sin^4(x) - \cos(x) \sin^6(x)\right) dx$$

$$= \frac{1}{5} \sin^5(x) - \frac{1}{7} \sin^7(x) + c$$

Problem: Find $\int \cos^8(x) \sin^5(x)\, dx$.

Sol^n: $\int \cos^8(x) \sin^5(x)\, dx$

$$= \int \cos^8(x) \sin(x) \sin^4(x)\, dx$$

use $\sin^2(x) = 1 - \cos^2(x)$

$$= \int \cos^8(x) \sin(x) \left(1 - \cos^2(x)\right)^2 dx$$

$$= \int \cos^8(x) \sin(x) \left(1 - 2\cos^2(x) + \cos^4(x)\right) dx$$

$$= \int \left(\cos^8(x) \sin(x) - 2\cos^{10}(x) \sin(x) + \cos^{12}(x) \sin(x)\right) dx$$

$$= -\frac{1}{9} \cos^9(x) + \frac{2}{11} \cos^{11}(x) - \frac{1}{13} \cos^{13}(x) + c.$$

<u>Note</u>: The last step is really a substitution:

$$\int \cos^{12}(x)\sin(x)\,dx \quad , \quad \text{use} \quad u = \cos(x)$$
$$du = -\sin(x)\,dx$$

$$= \int -u^{12}\,du = -\frac{1}{13}u^{13} + c = -\frac{1}{13}\cos^{13}(x) + c$$

<u>Problem</u>: Find $\int \cos^4(x)\sin^6(x)\,dx$

<u>Sol^n</u>: $\int \cos^4(x)\sin^6(x)\,dx$ use $\sin^2(x) = 1-\cos^2(x)$

$$= \int \cos^4(x)\left(1-\cos^2(x)\right)^3 dx$$

$$= \int \cos^4(x)\left(1 - 3\cos^2(x) + 3\cos^4(x) - \cos^6(x)\right)dx$$

$$= \int \left(\cos^4(x) - 3\cos^6(x) + 3\cos^8(x) - \cos^{10}(x)\right)dx$$

$$= \int \Bigg(\left(\tfrac{1}{2}(1+\cos(2x))\right)^2 - 3\left(\tfrac{1}{2}(1+\cos(2x))\right)^3$$
$$+ 3\left(\tfrac{1}{2}(1+\cos(2x))\right)^4 - \left(\tfrac{1}{2}(1+\cos(2x))\right)^5\Bigg)dx$$

it get tideous!

Maybe there is a pattern?

$$\int \cos^2(x)\,dx = \int \tfrac{1}{2}(1+\cos(2x))\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin(2x) + c$$

$$\int \cos^4(x)\,dx = \int \left(\tfrac{1}{2}(1+\cos(2x))\right)^2 dx$$

$$= \int \tfrac{1}{4}\left(1 + 2\cos(2x) + \cos^2(2x)\right)dx$$

$$= \int \left[\frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{4}\,\frac{1}{2}\left(1+\cos(4x)\right)\right] dx$$

$$= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{16}\sin(4x) + c$$

Not so obvious what the pattern should be!

Strategy for $\int \sin^k(x)\cos^m(x)\,dx$

① If m is odd: Save one $\cos(x)$ and replace the others using $\cos^2(x) = 1 - \sin^2(x)$. Then use the substitution $u = \sin(x)$

② If k is odd: Save one $\sin(x)$ and replace the others using $\sin^2(x) = 1 - \cos^2(x)$. Then use the substitution $u = \cos(x)$.

③ If both k and m are even, then [replace $\sin^2(x)$ by $1 - \cos^2(x)$ or $\cos^2(x)$ by $1 - \sin^2(x)$ and] use half angle equalities:

$$\cos^2(x) = \frac{1}{2}\left(1 + \cos(2x)\right)$$

$$\sin^2(x) = \frac{1}{2}\left(1 - \cos(2x)\right)$$

Now try again!

$$\int \cos^4(x)\sin^6(x)\,dx$$

$$= \int \left[\tfrac{1}{2}(1+\cos(2x))\right]^2 \left[\tfrac{1}{2}(1-\cos(2x))\right]^3 dx$$

$$= \int \tfrac{1}{4}\left(1 + 2\cos(2x) + \cos^2(2x)\right)\tfrac{1}{8}\left(1 - 3\cos(2x) + 3\cos^2(2x) - \cos^3(2x)\right) dx$$

$$= \frac{1}{32}\int\left(1 - \cos(2x) - 2\cos^2(2x) + 2\cos^3(2x)\right) dx$$

$$= \frac{1}{32}\int\left(1 - \cos(2x) - (1+\cos(4x)) + 2\cos(2x)(1-\sin^2(2x))\right)dx$$

$$= \frac{1}{32}\left(x - \tfrac{1}{2}\sin(2x) - x - \tfrac{1}{4}\sin(4x) + \int\left[2\cos(2x) - 2\cos(2x)\sin^2(2x)\right]dx\right.$$

$$= \frac{1}{32}\left(-\tfrac{1}{2}\sin(2x) - \tfrac{1}{4}\sin(4x) + \sin(2x) - \tfrac{1}{3}\sin^3(2x)\right) + c$$

Check: The derivative is

$$\frac{1}{32}\left(-\cos(2x) - \cos(4x) + 2\cos(2x) - 2\cos(2x)\sin^2(x)\right)$$