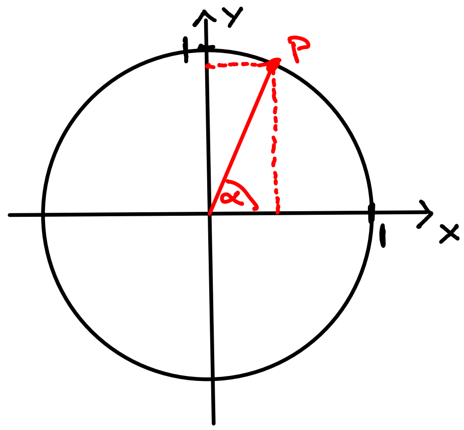
Trigonometric Integrals

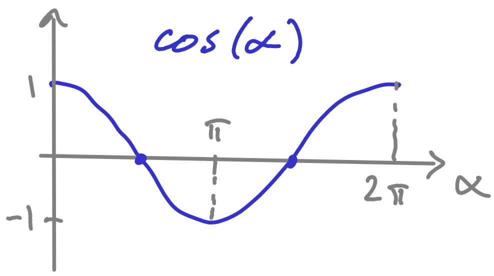


Unit circle.

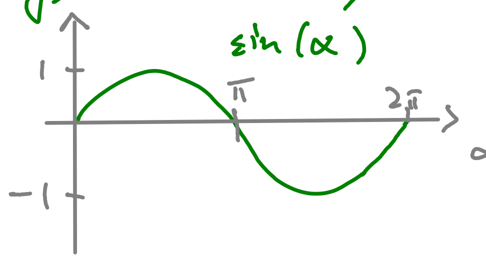
Measure angles in radians

$P = (\cos(\alpha), \sin(\alpha))$

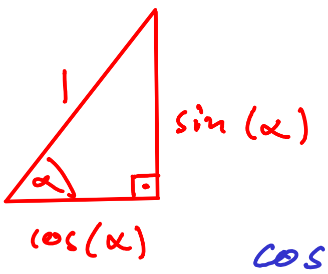
Cosine is the x-coordinate of a point on the unit circle.
When the point moves (with constant speed) anti-clockwise
starting at $(1,0)$, i.e. $\alpha = 0$, then we get



Similarly, sine is the y-coordinate



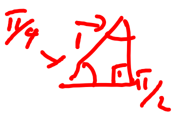
Identities:



$$\cos^2(\alpha) + \sin^2(\alpha) = 1$$

$\cos(\alpha) = 0$ at $\alpha = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$

i.e. $\alpha = \frac{2k+1}{2}\pi$ with $k \in \mathbb{Z}$



$\sin(\alpha) = 0$ at $\alpha = 0, \pm\pi, \pm 2\pi, \ldots$

i.e. $\alpha = k\pi$ with $k \in \mathbb{Z}$

Some common values:

α	0	$\pi/4$	$\pi/3$	$\pi/2$	π
$\cos(\alpha)$	1	$1/\sqrt{2}$	$1/2$	0	-1
$\sin(\alpha)$	0	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0

$\cos(\alpha) = \cos(-\alpha)$, i.e. cosine is an <u>even</u> function

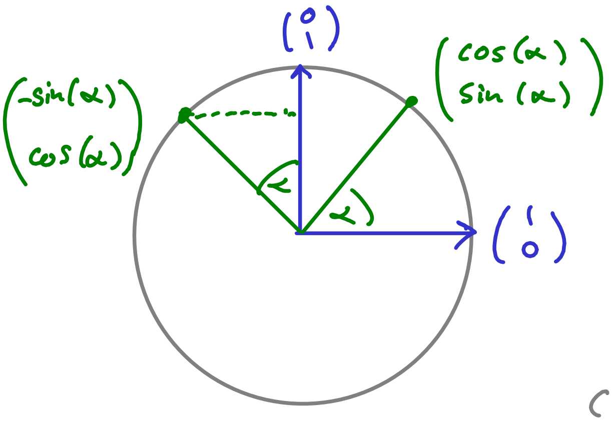
$\sin(\alpha) = -\sin(-\alpha)$, i.e. sine is an <u>odd</u> function.

$$\cos(\alpha) = \sin\left(\alpha + \frac{\pi}{2}\right)$$

<u>In general:</u> $A \sin(\omega t + \phi)$

amplitude (A) frequency (ω) phase (ϕ)

<u>Rotations:</u> A rotation about the origin by an angle α in the anti-clockwise direction can be represented by a 2×2 matrix



$$\rightsquigarrow R_\alpha = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

<u>Check:</u> $R_\alpha \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos(\alpha)\\ \sin(\alpha)\end{pmatrix}$

$$R_\beta R_\alpha = R_{\alpha+\beta} = \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & ? \\ \cos(\alpha)\sin(\beta) + \cos(\beta)\sin(\alpha) & ? \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

So we find the Addition Theorems for sine & cosine:

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha+\beta) = \cos(\alpha)\sin(\beta) + \cos(\beta)\sin(\alpha)$$

Setting $\alpha=\beta$ gives

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$$

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

The first one gives

$$\cos(2\alpha) = 2\cos^2(\alpha) - (\cos^2(\alpha) + \sin^2(\alpha))$$

$$= 2\cos^2(\alpha) - 1$$

So $\cos^2(\alpha) = \frac{1}{2}(1+\cos(2\alpha))$

Now find $\int \cos^3(x)\sin^4(x)\,dx$