Partial Fractions continued

Problem: Find $\displaystyle\int \frac{3\omega-1}{\omega+2}\,d\omega$

Observation: $\displaystyle\frac{d}{dx}\left(\ln(f(x)\right) = \frac{f'(x)}{f(x)}$

Examples:
- $\displaystyle\frac{d}{dx}\left(\ln(x+2)\right) = \frac{1}{x+2}$
- $\displaystyle\frac{d}{dx}\left(\ln(x^2+2x-3)\right) = \frac{2x+2}{x^2+2x-3}$

Solution: $\displaystyle\int \frac{3\omega-1}{\omega+2}\,d\omega = \int \frac{3(\omega+2)-7}{\omega+2}\,d\omega$

$$= \int\left(\frac{3(\omega+2)}{\omega+2} - 7\,\frac{1}{\omega+2}\right)d\omega$$

$$= \int 3\,d\omega - 7\int \frac{1}{\omega+2}\,d\omega$$

$$= 3\omega - 7\ln|\omega+2| + c$$

Check: $\displaystyle\frac{d}{d\omega}\left(3\omega - 7\ln|\omega+2|\right)$

$$= 3 - 7\,\frac{1}{\omega+2} = \frac{3\omega+6-7}{\omega+2} = \frac{3\omega-1}{\omega+2}$$

The above observation becomes

$$\ln(f(x)) = \int \frac{f'(x)}{f(x)}\,dx$$

when we integrate both sides.

Problem: Find $\displaystyle\int \frac{4x-4}{x^2-2x+5}\,dx$

Sol^n: $\displaystyle\int \frac{4x-4}{x^2-2x+5}\,dx = 2\int \frac{2x-2}{x^2-2x+5}\,dx$

$$= 2\ln|x^2-2x+5| + c$$

Problem: Find $\displaystyle\int \frac{x^3-1}{x^2+x}\,dx$

$$(x^3-1)/(x^2+x) = x-1+\frac{x-1}{x^2+x}$$

$$\begin{array}{l} x^3+x^2 \\ \hline -x^2-1 \\ -x^2-x \\ \hline \quad x-1 \end{array}$$

$$3146/15 = 29+\frac{11}{15}$$

$$\begin{array}{l} 30 \\ \hline 146 \\ 135 \\ \hline 11 \end{array}$$

Check: $\left(x-1+\frac{x-1}{x^2+x}\right)(x^2+x) = (x-1)(x^2+x)+x-1$

$$= x^3-x^2+x^2-x+x-1$$

$$= x^3-1$$

Now go back to the problem:

$$\int \frac{x^3-1}{x^2+x}\,dx = \int \left(x - 1 + \frac{x-1}{x^2+x} \right) dx$$

$$= \frac{1}{2}x^2 - x + \int \frac{x-1}{x^2+x}\,dx$$

$$= \frac{1}{2}x^2 - x + \int \frac{1}{2}\,\frac{2x+1-3}{x^2+x}\,dx$$

$$= \frac{1}{2}x^2 - x + \frac{1}{2}\int \left(\frac{2x+1}{x^2+x} - \frac{3}{x^2+x} \right) dx$$

$$= \frac{1}{2}x^2 - x + \frac{1}{2}\ln|x^2+x| - \frac{1}{2}\int \frac{3}{x^2+x}\,dx$$

Aside:

$$\frac{3}{x^2+x} = \frac{3}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$= \frac{(A+B)x + A}{x(x+1)}$$

$A = 3$ and $A+B=0$, so $B=-3$

$$= \frac{1}{2}x^2 - x + \frac{1}{2}\ln|x^2+x| - \frac{1}{2}\int \left(\frac{3}{x} - \frac{3}{x+1} \right) dx$$

$$= \frac{1}{2}x^2 - x + \frac{1}{2}\ln|x^2+x| - \frac{3}{2}\ln|x| + \frac{3}{2}\ln|x+1| + C$$