More integrals

Problem: Find $\int_0^2 x\sqrt{1+x}\,dx$.

Sol^n: Try a substitution: let $u = 1+x$ $= u(x)$. Then $du = dx$ and $x = u-1$. So

$3 = 1+2 = u(2)$
$1 = 1+0 = u(0)$

$$\int_0^2 x\sqrt{1+x}\,dx = \int_1^3 (u-1)\sqrt{u}\,du$$

$$= \int_1^3 \left(u^{3/2} - u^{1/2}\right) du = \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^3$$

$$= \frac{2}{5}\sqrt{3}^5 - \frac{2}{3}\sqrt{3}^3 - \left(\frac{2}{5} - \frac{2}{3}\right)$$

$$= \sqrt{3}\left(\frac{2}{5}\cdot 9 - \frac{2}{3}\cdot 3\right) + \frac{4}{15}$$

$$= \frac{1}{15}\left(24\sqrt{3} + 4\right)$$

Alternatively: Try integration by parts.

The formula $\int u\,dv = uv - \int v\,du$ (in the tables)

Preferably: $\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$

In our example:

$$\int_0^2 \underbrace{x}_{g}\,\underbrace{\sqrt{1+x}}_{f'}\,dx$$

$$\boxed{\begin{array}{ccc} & \int\sqrt{1+x}\,dx & = \frac{2}{3}(1+x)^{3/2} \\ u=1+x & \| & \| \\ & \int\sqrt{u}\,du & = \frac{2}{3}u^{3/2}\end{array}}$$

$$= x\,\frac{2}{3}(1+x)^{3/2} - \int \frac{2}{3}(1+x)^{3/2}\,dx$$

$$= \left(\frac{2}{3}x(1+x)^{3/2} - \frac{4}{15}(1+x)^{5/2}\right]_0^2 = \frac{4}{3}\sqrt{3}^3 - \frac{4}{15}\sqrt{3}^5 + \frac{4}{15}$$

$$= \frac{1}{15}\left(\sqrt{3}\,(20\cdot 3 - 36) + 4\right) = \frac{1}{15}\left(24\sqrt{3} + 4\right)$$

Partial Fractions Method

Problem: Find $\int \frac{1}{x(1-x)}\,dx$

Sol^n: Idea: Write $\frac{1}{x(1-x)}$ as a sum $\frac{A}{x} + \frac{B}{1-x}$.

From $\frac{A}{x} + \frac{B}{1-x} = \frac{A(1-x) + Bx}{x(1-x)} = \frac{x(B-A) + A}{x(1-x)}$,

if we want this to be $\frac{1}{x(1-x)}$, then we need $A = 1$ and $B - A = 0$. So $B = 1$ as well.

Hence $\int \frac{1}{x(1-x)}\,dx = \int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx$

$$= \int \frac{1}{x}\,dx + \int \frac{1}{1-x}\,dx,$$

Use $u = 1-x$. Then $du = -dx$

$$= \ln|x| - \int \frac{1}{u}\,du$$

$$= \ln|x| - \ln|u| = \ln|x| - \ln|1-x|$$

$$= \ln\left|\frac{x}{1-x}\right|$$

Check: $\frac{d}{dx}\left(\ln\left|\frac{x}{1-x}\right|\right) = \frac{1}{\frac{x}{1-x}} \cdot \frac{d}{dx}\left(\frac{x}{1-x}\right)$ chain rule

$$= \frac{1-x}{x} \cdot \frac{1-x+x}{(1-x)^2} = \frac{1}{x(1-x)}$$

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One more example: Find $\int \frac{2}{(x-1)(x-2)}\,dx$

Solve $\frac{A}{x-1} + \frac{B}{x-2} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)} = \frac{2}{(x-1)(x-2}$

for A and B.

$$= \frac{(A+B)x - 2A - B}{(x-1)(x-2)}$$

We need $A+B=0$ and $-2A-B=2$

$A=-B \implies -A=2$ and $B=+2$.

So $\int \frac{2}{(x-1)(x-2)}\,dx = \int \left(\frac{2}{x-2} - \frac{2}{x-1} \right) dx$

$$= 2\ln|x-2| - 2\ln|x-1| + c$$