More substitutions

Problem: Find $\int t\sqrt{1+2t^2}\,dt$

Sol^n: Let $u = 1+2t^2$. Then $\frac{du}{dt} = 4t$, or

$\frac{1}{4}\,du = t\,dt$. Now we get

$$\int \sqrt{1+2t^2}\; t\,dt = \frac{1}{4}\int \sqrt{u}\,du = \frac{1}{4}\int u^{1/2}\,du$$

$$= \frac{1}{4}\,\frac{2}{3}\,u^{3/2} + c = \frac{1}{6}\,(1+2t^2)^{3/2} + c$$

Problem: Find $\int \frac{1}{1+x^2}\,dx$.

Sol^n: Let $x = \tan(u)$. Then

$$\frac{dx}{du} = \frac{d}{du}\left(\frac{\sin(u)}{\cos(u)}\right) = \frac{\cos^2(u) - (-\sin(u))\sin(u)}{\cos^2(u)} = \frac{1}{\cos^2(u)}$$

and so $dx = \frac{1}{\cos^2(u)}\,du$. Next

$$\int \frac{1}{1+x^2}\,dx = \int \frac{1}{1+\tan^2(u)}\,\frac{1}{\cos^2(u)}\,du$$

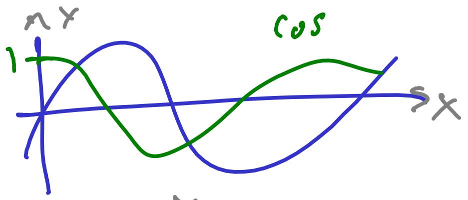
$$= \int \frac{1}{1+\frac{\sin^2(u)}{\cos^2(u)}} \cdot \frac{1}{\cos^2(u)}\,du$$

$$= \int \frac{1}{\cos^2(u)+\sin^2(u)}\,du = \int 1\,du = u + c$$

$$= \tan^{-1}(x) + c$$

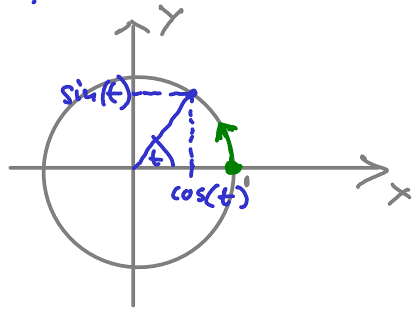
Problem: Find $\int \sqrt{1-x^2}\, dx$ using the substitution $x = \sin(u)$.

Sol^n: $\frac{dx}{du} = \cos(u)$



$$\int \sqrt{1-x^2}\, dx$$

$$= \int \sqrt{1-\sin^2(u)}\, \cos(u)\, du$$



$\sin^2(t) + \cos^2(t) = 1$

$$= \int \sqrt{\cos^2(u)}\, \cos(u)\, du$$

$$= \int \cos^2(u)\, du$$

$$= \frac{1}{2}\left(u + \frac{1}{2}\sin(2u)\right) + c$$

How did this happen? We used the tables.

$$= \frac{1}{2}\left(\sin^{-1}(x) + \frac{1}{2}\sin\left(2\sin^{-1}(x)\right)\right) + c$$

Let's see if we can do this ourselves.

$$\int \cos^2(u)\, du$$

$$= \int \frac{1}{2}(\cos(2u)+1)\, du$$

$$= \int \frac{1}{2} + \frac{1}{2}\cos(2u)\, du$$

$$= \frac{1}{2}u + \frac{1}{4}\sin(2u)$$

Use the addition theorem for cosine.

$$\cos(2u) = \cos^2(u) - \sin^2(u)$$
$$= 2\cos^2(u) - (\cos^2(u) + \sin^2(u))$$
$$= 2\cos^2(u) - 1$$

So $\cos^2(u) = \frac{1}{2}(\cos(2u)+1)$

another substitution $v = 2u$