The Net Change Theorem

Theorem: The integral of a rate of change $F'(x)$ from a to b is the net chang of $F(x)$,

i.e. $$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Examples: ① If water flows out of tank at a rate $\frac{dV}{dt}$ then $$\int_{t_0}^{t_1} \frac{dV}{dt}\,dt = V(t_1) - V(t_0)$$

② If $N(t)$ is the size of a population which grows at the rate $\frac{dN}{dt}$, then

$$\int_2^{20} \frac{dN}{dt}\,dt = N(20) - N(2)$$

③ Suppose a particle has velocity $v(t)$ at time t, then $\int_{t_0}^{t_1} v(t)\,dt$ = displacement of the particle from time t_0 to time t_1

If you want the total distance that particle covered between t_0 and t_1, then you need to calculate

$$\int_{t_0}^{t_1} |v(t)|\,dt$$

More specific examples:

(1) The rate of change of charge of a capacitor (an electronic component that stores charge) is

$$\frac{dC}{dt}(t) = 1 - e^{-5t}$$

What is the net change of charge between $t_0 = 1$ and $t_1 = 5$?

Sol^n: The answer $C(5) - C(1)$ which is equal to

$$\int_1^5 \frac{dC}{dt}\,dt = \int_1^5 (1 - e^{-5t})\,dt$$

$$= \left[t + \frac{1}{5} e^{-5t} \right]_1^5$$

$$= 5 + \frac{1}{5} e^{-25} - \left(1 + \frac{1}{5} e^{-5}\right)$$

$$= 4 + \frac{1}{5}\left(e^{-25} - e^{-5}\right)$$

$$\frac{d}{dt}\left(e^{-5t}\right) = -5e^{-5t}$$

Why? Chain rule!

$u = -5t$

$g = e^u$

$$\frac{dg}{dt} = \frac{dg}{du}\,\frac{du}{dt}$$

$$= e^u \cdot (-5)$$

$$= e^{-5t}(-5) = -5e^{-5t}$$

Method of substitution

Problem: Find $\int x e^{x^2}\, dx$

Let's write $u = x^2$. Then $\frac{du}{dx} = 2x$ and we get

$du = 2x\, dx$, or $\frac{1}{2} du = x\, dx$.

Then $\int x e^{x^2} dx = \int e^{x^2} x\, dx = \int e^u \frac{1}{2} du$

$$= \frac{1}{2} \int e^u\, du = \frac{1}{2} e^u + c = \frac{1}{2} e^{x^2} + c, \quad c = \text{const.}$$

Check: $\frac{d}{dx}\left(\frac{1}{2} e^{x^2} + c\right) = \frac{1}{2} e^{x^2} \frac{d}{dx}(x^2)$ Chain Rule

$$= \frac{1}{2} e^{x^2} 2x = x e^{x^2}$$