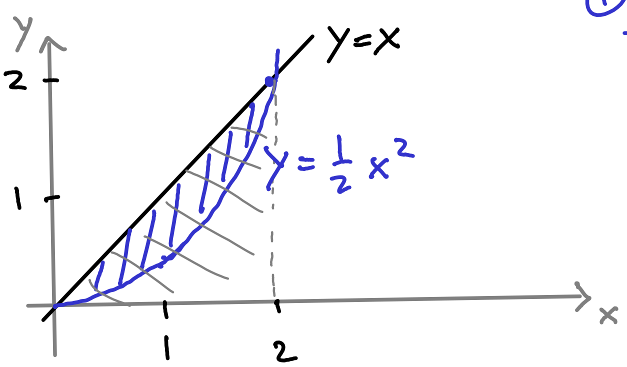
Area between curves

Problem: Calculate the area between the graphs of $y = x$ and $y = \frac{1}{2}x^2$.



① Where do the graphs meet?

Set the two equations/expressions in x equal:

$$x = \frac{1}{2}x^2 \quad \text{or}$$

$$0 = x^2 - 2x = x(x-2)$$

So $x = 0$ or $x = 2$.

② Find out which curve is the upper bound.

Here it is $y = x$

③ The area is the integral of (upper − lower) from the left point of intersection to the right point of intersection.

$$\text{So} \int_0^2 \left(x - \frac{1}{2}x^2\right) dx = \left[\frac{1}{2}x^2 - \frac{1}{6}x^3\right]_0^2 = 2 - \frac{4}{3} - (0) = \frac{2}{3}.$$

Problem: Let $f(x) = x^2 - x$ and $g(x) = x + 1$ and find the area they enclose.

① Points of intersection: Solve $x^2 - x = x + 1$ or

$$x^2 - 2x - 1 = 0$$

$$x = 1 \pm \sqrt{2}$$

② Which function is the upper bound? Plug in a value between the points of intersection, say 1 in this case

$$f(1) = 0 < g(1) = 2$$

③ So the area is

$$\int_{1-\sqrt{2}}^{1+\sqrt{2}} x+1-(x^2-x)\,dx = \int_{1-\sqrt{2}}^{1+\sqrt{2}} \underline{(2x+1-x^2)}\,dx$$

$$= \left[x^2 + x - \frac{1}{3}x^3 \right]_{1-\sqrt{2}}^{1+\sqrt{2}}$$

$$= 3+2\sqrt{2} + 1+\sqrt{2} \underline{- \frac{1}{3}(1+\sqrt{2})^3} - \left(3-2\sqrt{2} + 1 - \sqrt{2} - \frac{1}{3}(1-\sqrt{2})^3\right)$$

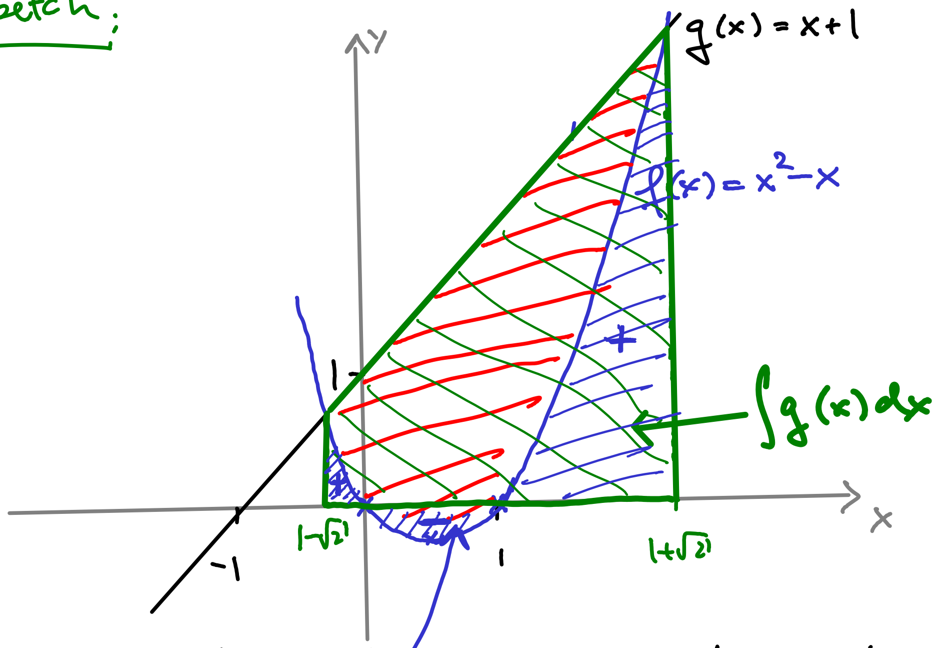
$$= 4\sqrt{2} + 2\sqrt{2} - \frac{1}{3}\left((1+\sqrt{2})^3 - (1-\sqrt{2})^3\right)$$

$$= 6\sqrt{2} - \frac{1}{3}\left(1+3\sqrt{2}+6+2\sqrt{2} - \left(1-3\sqrt{2}+6-2\sqrt{2}\right)\right)$$

$$\left[\text{use } (a+b)^3 = a^3 + \underline{3}ab^2 + \underline{3}a^2b + b^3\right]$$

$$= 6\sqrt{2} - \frac{1}{3}\left(6\sqrt{2} + 4\sqrt{2}\right) = \sqrt{2}\left(6 - \frac{10}{3}\right) = \frac{8}{3}\sqrt{2}.$$

Sketch:



$f(x) = x^2 - x$
$= x(x-1)$

$f(\frac{1}{2}) = -\frac{1}{4}$

Since area below the x-axis counts negatively, when we subtract $\int f(x)\,dx$ we actually add the area that's below the x-axis. So we get indeed the correct answer!