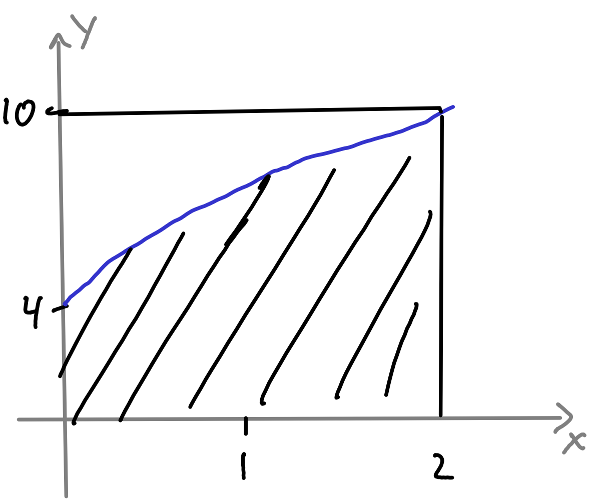
Some more examples of integrals

Calculate $\int_0^2 (x^3 - 2x^2 + 3x + 4)\,dx$ and interpret it as an area.

$$\int_0^2 (x^3 - 2x^2 + 3x + 4)\,dx = \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{3}{2}x^2 + 4x\right]_0^2$$

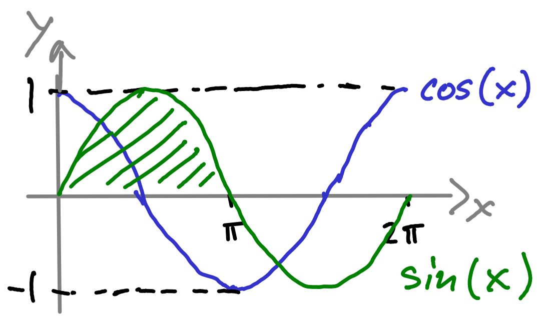
$$= 4 - \frac{16}{3} + 6 + 8 - (0) = 18 - \frac{16}{3} = \frac{38}{3}$$

It is the shaded area!



Evaluate $\int_0^{\pi} \sin(x)\,dx$.

<u>Sol^n</u>: $\int_0^{\pi} \sin(x)\,dx = \left[-\cos(x)\right]_0^{\pi} = 1 - (-1) = 2$



Obviously $\int_0^{2\pi} \sin(x)\,dx = 0$ (area below x-axis counts as negative area)

Integration by parts

Recall the product rule for derivatives:

Let f and g be functions, then

$$(fg)' = f'g + fg'.$$

Integrating both sides gives

$$\int (fg)'\,dx = \int (f'g + fg')\,dx = \int f'g\,dx + \int fg'\,dx$$

$$\int (fg)'\,dx = fg$$

Consequently: $\displaystyle \int f'g\,dx = fg - \int fg'\,dx$

Example: Find $\displaystyle\int \underbrace{x}_{g}\,\underbrace{e^x}_{f'}\,dx$!

Sol^n: Let $g(x) = x$ and $f'(x) = e^x$. Then $g'(x) = 1$ and $f(x) = e^x$ and we get

$$\int xe^x\,dx = \int (g f')\,dx = fg - \int fg'\,dx = xe^x - \int e^x\,dx$$

$$= xe^x - e^x + C$$

Check: $\dfrac{d}{dx}(xe^x - e^x) = 1e^x + xe^x - e^x = xe^x$

Find $\int \ln(x)\,dx$ using integration by parts.

Sol^n: $\int \ln(x)\,dx = \int \underbrace{1}_{f'} \cdot \underbrace{\ln(x)}_{g}\,dx = x\ln(x) - \int x \frac{1}{x}\,dx$

$= x\ln(x) - \int 1\,dx = x\ln(x) - x + C$

Find $\int x\cos(x)\,dx$.

Sol^n: $\int \underbrace{x}_{g} \underbrace{\cos(x)}_{f'}\,dx = \underbrace{x}_{g}\underbrace{\sin(x)}_{f} - \int \underbrace{1}_{g'} \cdot \underbrace{\sin(x)}_{f}\,dx$

$= x\sin(x) + \cos(x) + C$

Find $\int x^3\cos(x)\,dx$.

Sol^n: $\int \underbrace{x^3}_{g}\underbrace{\cos(x)}_{f'}\,dx = x^3\sin(x) - \int 3\underbrace{x^2}_{u}\underbrace{\sin(x)}_{v'}\,dx$

$= x^3\sin(x) - 3\left[-x^2\cos(x) - \int 2x(-\cos(x))\,dx\right]$

$= x^3\sin(x) + 3x^2\cos(x) - 6\int x\cos(x)\,dx$

$= x^3\sin(x) + 3x^2\cos(x) - 6x\sin(x) - 6\cos(x) + C$

Check: $3x^2\sin(x) + x^3\cos(x) + 6x\cos(x) - 3x^2\sin(x)$

$- 6\sin(x) - 6x\cos(x) + 6\sin(x) = x^3\cos(x)$