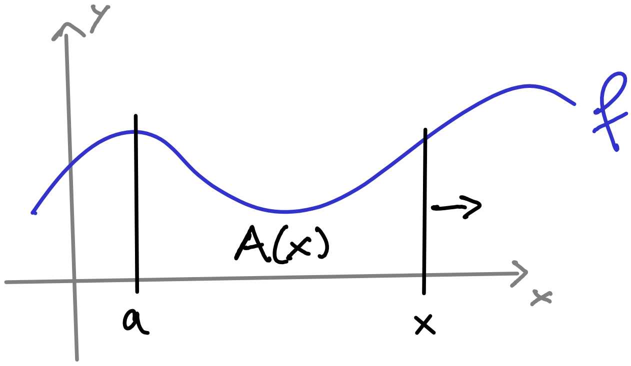
Fundamental Theorem of Calculus

let's make the right hand boundary of an integral variable.



$$A(x) = \int_a^x f(t)\,dt$$

Intuitively, it seems that $\frac{dA}{dx}$, the rate of change of the area $A(x)$, depends on $f(x)$.

Fundamental Theorem of Calculus:

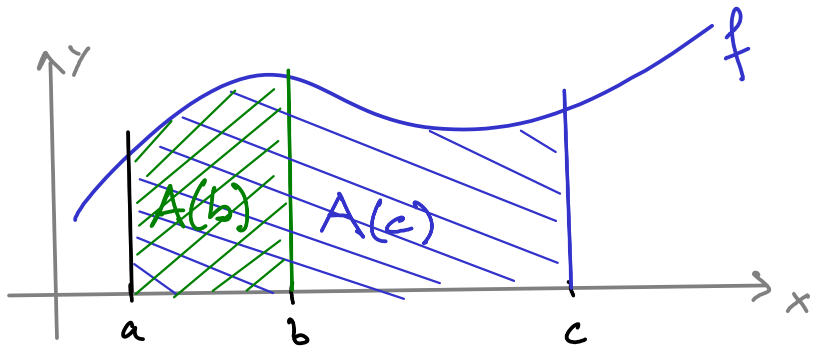
If $f(x)$ is continuous, and we define

$$A(x) = \int_a^x f(t)\,dt \quad , \quad \text{then}$$

$\frac{dA}{dx}(x) = f(x)$. In other words, $A(x)$ is an antiderivative of $f(x)$.

Consequences: If $B(x) = \int_b^x f(t)\,dt$, then $B(x)$ is an antiderivative of f, and A and B differ by the constant $\int_a^b f(t)\,dt$.

(2) $$\int_b^c f(t)\,dt = A(c) - A(b)$$



Applications

(1) Find the derivative of $F(x) = \int_a^x \sqrt{1+t^2}\,dt$.

By the Fundamental Theorem of Calculus, it is simply $\frac{dF}{dx}(x) = \sqrt{1+x^2}$

(2) Find the derivative of $F(x) = \int_a^{x^4} \sqrt{1+t^2}\,dt$

Make a substitution $z = x^4$, then look at

at
$$F(x) = F(z(x)) = \int_a^z \sqrt{1+t^2}\,dt$$

Now $\frac{dF}{dx} = \frac{dF}{dz}\frac{dz}{dx}$ by the chain rule.

The FTC says that $\frac{dF}{dz} = \sqrt{1+z^2}$ and $\frac{dz}{dx} = 4x^3$,

so $\frac{dF}{dx}(x) = 4x^3\sqrt{1+x^8}$.

Upshot: Substitute x^4 for t and multiply by

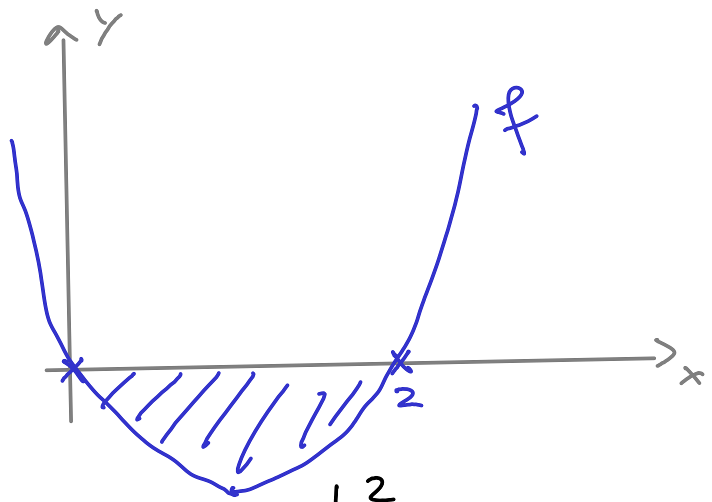
$$\frac{d}{dx}(x^4) = 4x^3.$$

One more: Find $\frac{dF}{dx}$ if $F(x) = \int_a^{3x} \sin(e^t)\,dt$.

Just like above:

$$\frac{dF}{dx}(x) = 3\sin(e^{3x})$$

(3) Find the area under the graph of $f(x) = x^2 - 2x$ that is below the x-axis.



So the area is $\left| \int_0^2 (x^2 - 2x)\,dx \right|$. We know that

$F(x) = \frac{1}{3}x^3 - x^2$ is an antiderivative of $x^2 - 2x$

and so

$$\int_0^2 (x^2 - 2x)\,dx = F(2) - F(0) = \frac{1}{3}2^3 - 4 - 0$$

$$= \frac{8}{3} - 4 = \frac{-4}{3}$$ and the area is $\frac{4}{3}$.