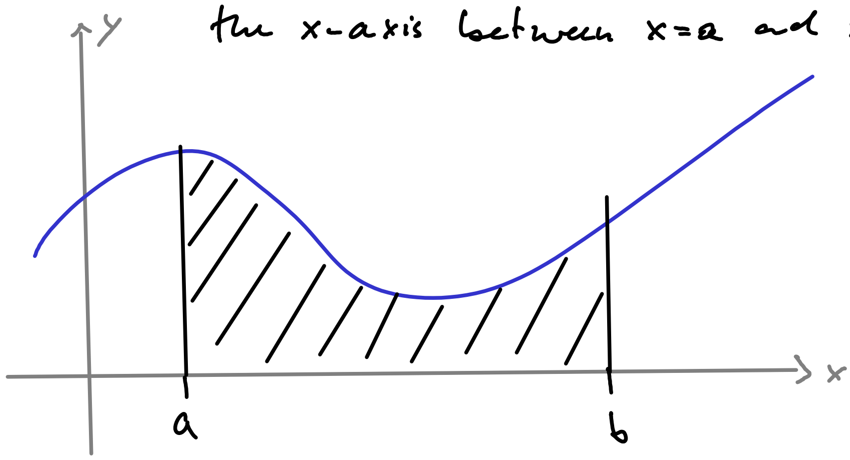
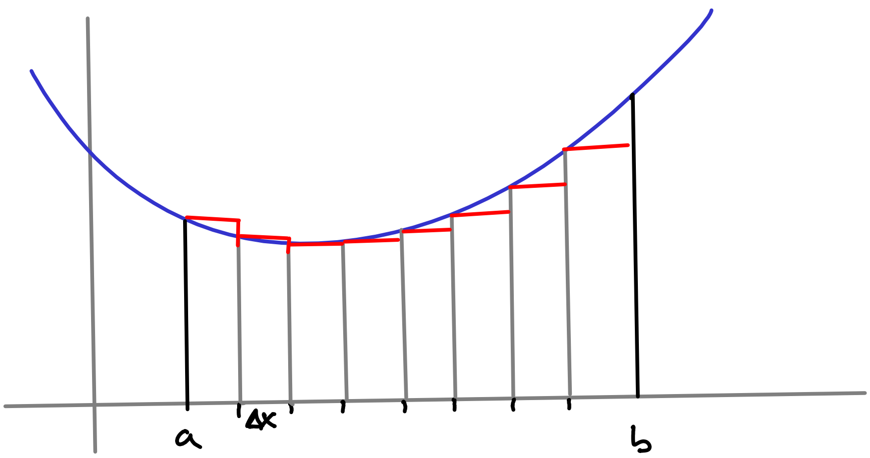
The Area Problem

Suppose $f : \mathbb{R} \to \mathbb{R}$ with $f(x) \geq 0$ for all $x \in \mathbb{R}$
Fix $a, b \in \mathbb{R}$. What is the are below the graph of f and above the x-axis between $x=a$ and $x=b$?



The idea: Approximate the area by rectagles of a fixed width.



This approximation is obtained, using n rectagles each of width $\Delta x = \frac{b-a}{n}$ as

$$f(a)\Delta x + f(a+\Delta x)\Delta x + f(a+2\Delta x)\Delta x + \ldots + f(a+(n-1)\Delta x)\Delta x$$

$$= \left[f(a) + f(a+\Delta x) + \ldots + f(a+(n-1)\Delta x) \right] \Delta x$$

$$= \Delta x \sum_{k=0}^{n-1} f(a+k\Delta x)$$

Definition: We define the (definite) integral of $f:\mathbb{R}\to\mathbb{R}$ from a to b as the limit

$$\lim_{n\to\infty} \Delta x \sum_{k=0}^{n-1} f(a+k\Delta x), \text{ where } \Delta x = \frac{b-a}{a}.$$

The short notation for this limit, if it exists is

$$\int_a^b f(x)\,dx.$$

Properties of integrals $(b>a)$

① If f is continuous, then $\int_a^b f(x)\,dx$ exists.

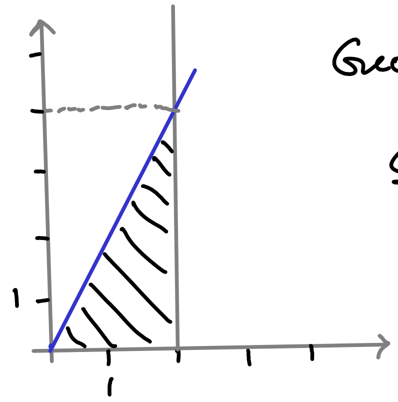
② $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

③ If $f \geq 0$, then $\int_a^b f(x)\,dx \geq 0$

If $f \leq 0$, then $\int_a^b f(x)\,dx \leq 0$

④ $$\int_a^b C f(x)\,dx = C \int_a^b f(x)\,dx$$

Example: Let $f(x) = 2x$ and find $\int_0^2 f(x)\,dx$



Geometrically, the answer is 4.

Some approximations:

<u>$n=2$</u>: $\Delta x = 1 = \frac{2-0}{2}$

$1 f(0) + 1 f(1) = 0 + 2 = 2.$

<u>$n=3$</u>: $\Delta x = \frac{2}{3}$: $\frac{2}{3}\left(f(0) + f\left(\frac{2}{3}\right) + f\left(\frac{4}{3}\right)\right)$

$$= \frac{2}{3}\left(0 + \frac{4}{3} + \frac{8}{3}\right) = \frac{24}{9}$$

$n=n$: $\Delta x = \frac{2}{n}$, $\frac{2}{n}\sum_{k=0}^{n-1} f\left(k\frac{2}{n}\right) = \frac{2}{n}\sum_{k=0}^{n-1}\frac{4k}{n}$