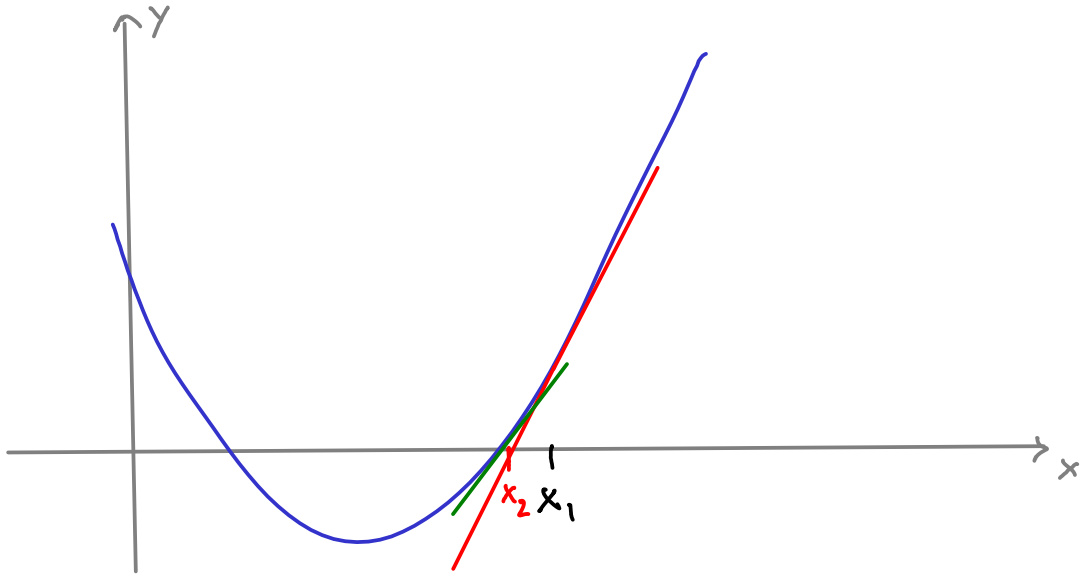
Calculus MA161/MA160 Sem II

Newton's Method

Consider a function $f(x)$ and the problem of finding a zero (or root) of it, that is a number a s.t. $f(a) = 0$.

For example $f(x) = x^2 - 2$. Can we approximate the solutions of $x^2 - 2 = 0$?



Idea: Find a first approximation x_1 (for example using the Intermediate Value Theorem)

① Then find the tangent to the graph of f at $(x_1, f(x_1))$.

② Use the x-axis intercept of this tangent as the next approximation.

Then repeat ① and ②

A bit of maths tells us that the equation for the tangent at x_1 is

$$t(x) = f'(x_1)\,x + c \quad \text{s.t.}$$

$$t(x_1) = f(x_1)$$

So $c = f(x_1) - f'(x_1)\,x_1$ and

$$t(x) = f'(x_1)\,(x - x_1) + f(x_1).$$

Hence $t(x) = 0$ implies

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Newton's formula is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$, i.e. the next approximation x_{i+1} is given in terms of the last approximation x_i.

Example: $f(x) = x^2 - 2$. So $f'(x) = 2x$

Choose $x_1 = 2$ as first approximation. Then

$$x_2 = 2 - \frac{2}{4} = \frac{3}{2} = 1.5$$

$$x_3 = \frac{3}{2} - \frac{1}{12} = \frac{17}{12} = 1.4166$$

$$x_4 = 1.4142157$$