Solution to the MA140 Mid-Term Exam 2012

Q1 (a) $\lim\limits_{x\to 0} \dfrac{x-\sin(x)}{x(x+2)} = \lim\limits_{x\to 0} \dfrac{1-\cos(x)}{2x+2} = \dfrac{1-1}{2} = 0$

by L'Hospital's Rule, as $x-\sin(x) \xrightarrow[x\to 0]{} 0$ and $x(x+2) \xrightarrow[x\to 0]{} 0$

Alternatively, $\lim\limits_{x\to 0} \dfrac{x-\sin(x)}{x(x+2)} = \lim\limits_{x\to 0} \dfrac{1}{x+2}\left(1-\dfrac{\sin(x)}{x}\right) = 0$, as $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x} = 1$.

(b) $\lim\limits_{n\to\infty} \dfrac{(n-2)(2n+3)}{n^2+n-6} = \lim\limits_{n\to\infty} \dfrac{2n^2-n-6}{n^2+n-6} = \lim\limits_{n\to\infty} \dfrac{2-\frac{1}{n}-\frac{6}{n^2}}{1+\frac{1}{n}-\frac{6}{n^2}} = 2$

Q2 (a) From $\ln(f(x)) = 2\ln(x-1) + 3\ln(x+2) - 2\ln(x+1) - \ln(x-4)$

and $\dfrac{d}{dx}\big(\ln(f(x))\big) = \dfrac{2}{x-1} + \dfrac{3}{x+2} - \dfrac{2}{x+1} - \dfrac{1}{x-4} = \dfrac{f'(x)}{f(x)}$, we

get $f'(x) = \dfrac{(x-1)^2(x+2)^3}{(x+1)^2(x-4)}\left(\dfrac{2}{x-1} + \dfrac{3}{x+2} - \dfrac{2}{x+1} - \dfrac{1}{x-4}\right)$.

(b) For $f(x) = \cos^2(x^3-3x^2+5x-3)$, using the chain rule,

$f'(x) = -2(3x^2-6x+5)\cos(x^3-3x^2+5x-3)\sin(x^3-3x^2+5x-3)$.

Q3 Here $f(x) = 9xe^x$. So $f'(x) = 9e^x(x+1)$ and $f''(x) = 9xe^x(x+2)$.

(a) x-intercept: $f(x) = 0 \iff x = 0$, since $9e^x > 0$ for all x.

So $(0,0)$ is the x-intercept, and also the y-intercept $[f(0)]$.

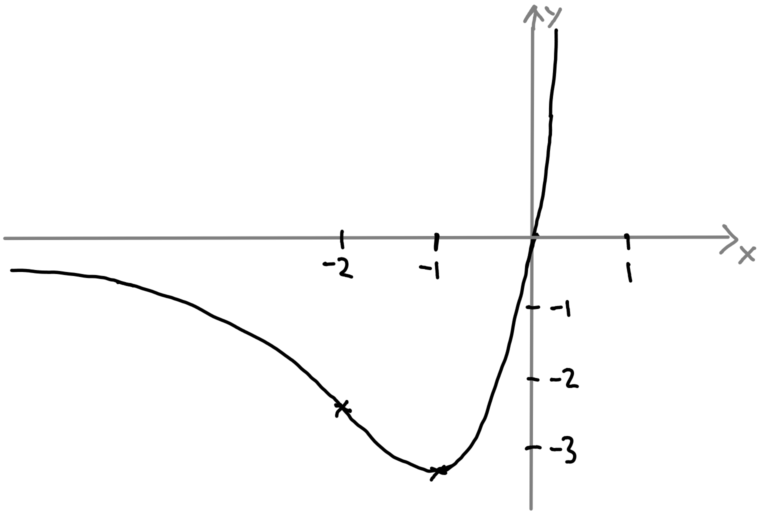
(b) $f'(x) = 0 \iff x = -1$

Since $f''(-1) = 9/e > 0$, it follows that f has a local minimum at $(-1, -9/e) = (-1, -3.311)$

(c) $f''(x) = 0 \iff x = -2$ (and since $f''(x) < 0$ for $x < -2$ and $f''(x) > 0$ for $x > 2$)

f has a point of inflection at $(-2, -18/e^2) = (-2, -2.436)$

(d)



(e) It has one asymptote, the x-axis.

Q4 (a) The volume of the tank is $V = \pi r^2 h = 16\pi h$,

so $dV = 16\pi\, dh = 16\pi \times 0.08 = 4.02$ and so the oil is pumped in at a rate of 4.02 cubic metre per minute.

(b) We know that $f(x) = Ae^{kt}$ and

$$Ae^{1200k} = f(1200) = \frac{1}{2} f(0) = A/2 .$$

So $1200k = \ln(1/2)$ and $k = \frac{-\ln(2)}{1200}$.

Then $f(100) = Ae^{\frac{-\ln(2)}{1200}} = 0.944\,A$, and hence 94.4% of the material is left after 100 years.

Q5 Let $f(x) = x^3 - x - 4$. Then $f'(x) = 3x^2 - 1$ and $f'(-1) = 2$ is the slope of the tangent line, which is given by $t(x) = 2x + b$ such that $t(-1) = f(-1)$, i.e. $-2 + b = -4$.

So $b = -2$ and $t(x) = 2x - 2$ is the equation of the tangent line.

Now solve $t(x) = f(x)$, i.e. $x^3 - 3x - 2 = 0$.

However, $x^3 - 3x - 2 = (x+1)(x^2 - x - 2) = 0$ holds if

$x = -1$ or $x = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2} = \frac{1 \pm 3}{2} = \begin{cases} -1 \\ 2 \end{cases}$. Hence the tangent and the graph of f intersect at $(-1, -4)$ and $(2, 2)$.