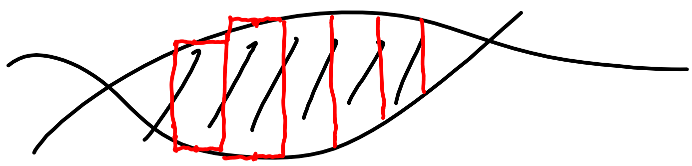
MA 140 Eng. Calculus

Quick introduction to limits.

This course is about differentiation and integration, both concepts involve limits

Integration: Problem is to calculate area between curves



Idea is to estimate and improve the estimate.

Use rectangles to approximate the area and then make those rectangles slimmer and slimmer and take the limit.

Differentiation (rates of change)

Suppose the velocity of a sky diver is

$$v(t) = 5t^2$$

What's the average speed between $t=2$ and $t=4$?

$$\frac{v(2)+v(4)}{2} = \frac{5\cdot 4 + 5\cdot 16}{2} = 5(2+8) = 50$$

What is the rate of change of speed at time $t = 3$?

$$\frac{dv}{dt}(3) = 2 \cdot 5 \cdot 3 = 30, \quad \frac{dv}{dt}(t) = 10t$$

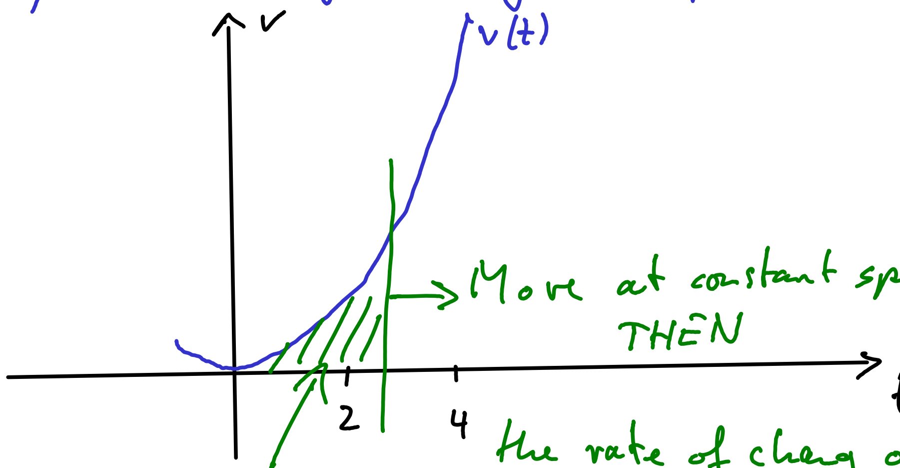
Fixing yesterdays calculations

We were given speed $v(t) = 5t^2$, and asked for average speed between $t=2$ and $t=4$

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time used}}$$

So for a solution, we need to get the distance travelled between $t=2$ and $t=4$!

velocity = rate of change of position



Move at constant speed THEN the rate of chang of this area is v(t)!

Solution: let $x(t)$ be the position of the skydiver. Then the rate of change of $x(t)$, that is $\frac{dx}{dt}(t)$ equals $v(t)$. In other words

$$x(t) = \int v(t)\,dt = \int 5t^2\,dt = \frac{5}{3}t^3 + c,$$

$c = \text{const.}$

$$\text{average speed} = \frac{x(4) - x(2)}{2} = \frac{5}{6}(4^3 - 2^3) = \frac{5}{3} \times 28 \approx 46.67$$

Secondly, we were asked the rate of chage of $v(t)$ at time $t=3$. We did that one fine.

Limits in more detail

Limits are taken of sequences of numbers, but not all sequences have limits

Example: Consider $f(x) = \dfrac{x^2-1}{x-1}$. This function is not defined at $x=1$. But we can ask what happens when x gets close to 1. We write

$\lim\limits_{x \to 1} f(x)$ to indicate that we let x approach 1 but never $x=1$.

Here we can do trick:

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} \overset{x \neq 1}{=} x+1 \xrightarrow[x \to 1]{} 2$$

which says

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$$

Example: let $f(x) = \dfrac{x+1}{x^2-1}$, which is not defined at -1 and 1. So what are

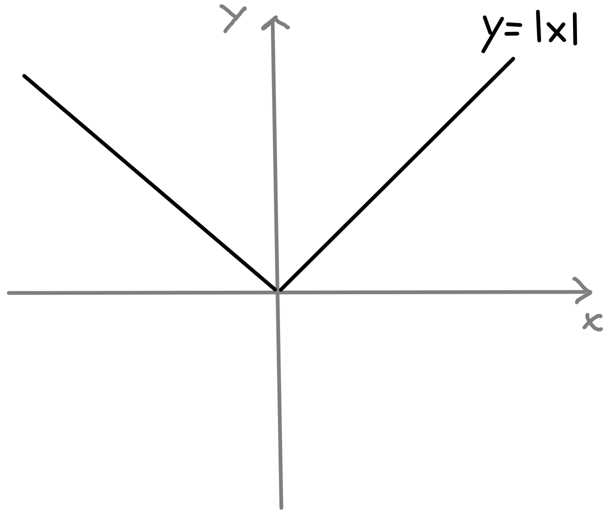
$\lim\limits_{x \to 1} f(x)$ and $\lim\limits_{x \to -1} f(x)$??

Again, $\dfrac{x+1}{x^2-1} = \dfrac{x+1}{(x+1)(x-1)} \overset{x \neq -1}{=} \dfrac{1}{x-1}$ $\xrightarrow{x \to -1} -\frac{1}{2}$, $\xrightarrow[x \to 1]{}$ does not exist

More on limits

The absolute value

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



Evaluate $\displaystyle\lim_{x \to 0} \frac{x}{|x-1| - |x+1|}$

For $x \neq 0$ but x close to 0, we have

$$\frac{x}{|x-1| - |x+1|} = \frac{x}{(1-x)-(x+1)} = \frac{x}{1-x-x-1} = \frac{x}{-2x} = -\frac{1}{2}$$

So $\displaystyle\lim_{x \to \infty} \frac{x}{|x-1| - |x+1|} = -\frac{1}{2}$

Limit definition

Limits are defined for sequences of number and they may or may not exist.

A sequence is an infinite ordered collection of numbers. Ex. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

write this as $a_n = \frac{1}{n}$, $n \in \mathbb{N}$

We say that the sequence $(a_n)_{n\in\mathbb{N}}$ has limit L if for all sufficiently large n we have a_n close to L.

In maths speak: For every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ s.t. for all $n \geq N$ $\underbrace{|a_n - L|}_{\text{this is distance between } a_n \text{ and } L} < \varepsilon$

In this case we write $\lim\limits_{n\to\infty} a_n = L$.

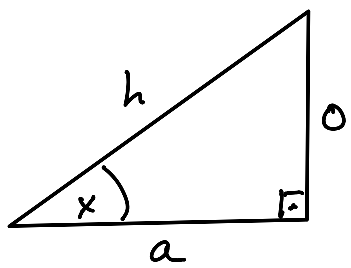
Ex: • $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

• $\lim\limits_{n\to\infty} \frac{2n^2+4n}{n^3+6} = 0$ Trick: $\frac{2n^2+4n}{n^3+6} \cdot \frac{1/n^3}{1/n^3}$

$$= \frac{2/n + 4/n^2}{1 + 6/n^3} \xrightarrow{n\to\infty} \frac{0+0}{1+0} = 0$$

• $\lim\limits_{n\to\infty} \frac{3n^2+6n-2}{4n^2-7n+3} = \lim\limits_{n\to\infty} \frac{3 + 6/n - 2/n^2}{4 - 7/n + 3/n^2} = \frac{3}{4}$

Limits of trigonometric functions



For $0 \leq x \leq \frac{\pi}{2}$ $(90°)$

$\sin(x) = \frac{o}{h}$ $\tan(x) = \frac{o}{a}$

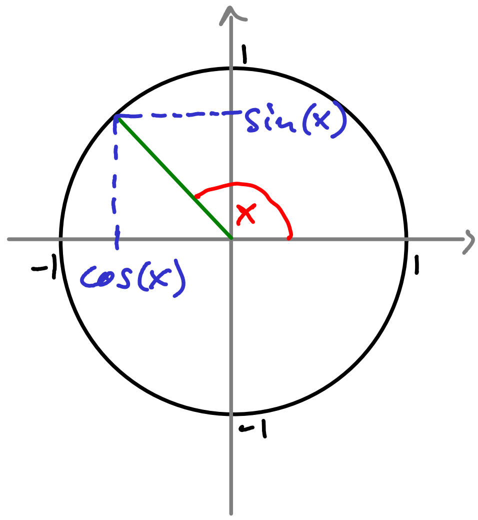
$\cos(x) = \frac{a}{h}$ $= \frac{\sin(x)}{\cos(x)}$

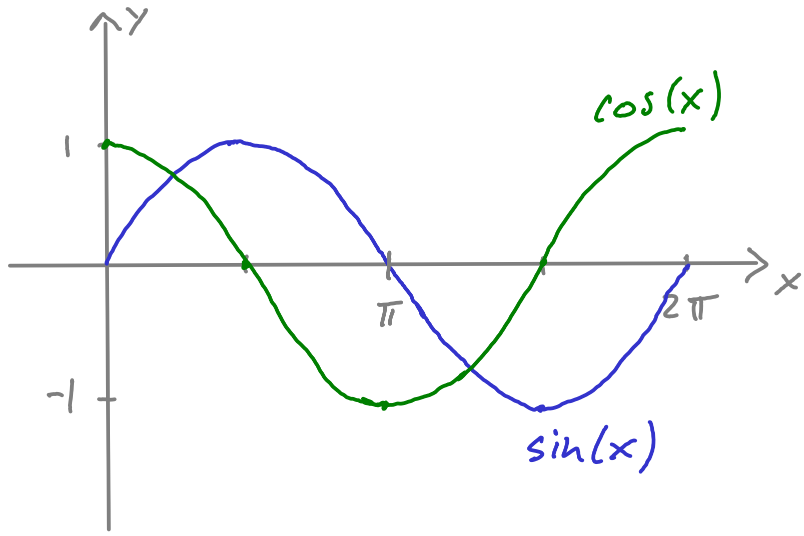
Useful identities: $\cos^2(x) + \sin^2(x) = 1$

$$\frac{a^2}{h^2} + \frac{o^2}{h^2} = \frac{a^2+o^2}{h^2} \underset{\text{Pythagoras}}{=} \frac{h^2}{h^2} = 1$$

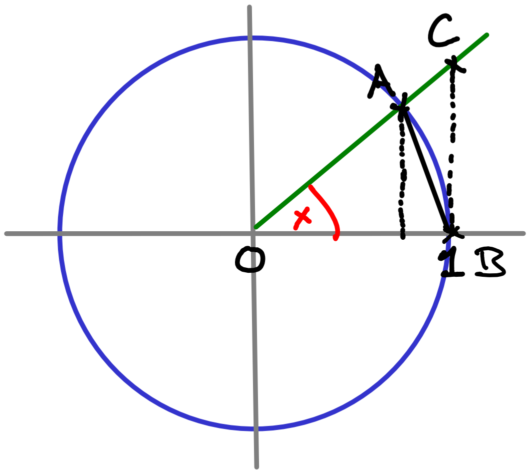
$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

Circle interpretation of sine and cosine





Limits continued



Problem: $\lim_{x \to 0} \frac{\sin(x)}{x} = ?$

For small x: Note that

area of triangle OAB $\leq$ area of sector OAB $\leq$ area of triangle OBC

So

$$\frac{1}{2}\sin(x) \leq \frac{1}{2}x \leq \frac{1}{2}\tan(x) = \frac{1}{2}\frac{\sin(x)}{\cos(x)}.$$

So $\sin(x) \leq x \leq \frac{\sin(x)}{\cos(x)}$

or $\frac{1}{\sin(x)} \geq \frac{1}{x} \geq \frac{\cos(x)}{\sin(x)}$

Note that for small $x>0$, we have $\sin(x) > 0$.

So $1 = \frac{\sin(x)}{\sin(x)} \geq \frac{\sin(x)}{x} \geq \cos(x)$

Now, as $x \to 0$, we have $\cos(x) \to 1$.

Hence

$$\boxed{\lim_{x \to 0} \frac{\sin(x)}{x} = 1}$$

Note: We should consider $x < 0$ and small as well.

Example: Evaluate $\displaystyle\lim_{x\to 0} \frac{\cos(x)-1}{x}$.

Solution:
$$\frac{\cos(x)-1}{x} = \frac{\cos(x)-1}{x}\,\frac{\cos(x)+1}{\cos(x)+1}$$

$$= \frac{\cos^2(x)-1}{x\,(\cos(x)+1)} = \frac{-\sin^2(x)}{x\,(\cos(x)+1)} = \frac{-\sin(x)}{x}\,\frac{\sin(x)}{\cos(x)+1}$$

So
$$\lim_{x\to 0}\frac{\cos(x)-1}{x} = \lim_{x\to 0} \overbrace{\frac{-\sin(x)}{x}}^{\to 1}\;\overbrace{\frac{\sin(x)}{\cos(x)+1}}^{\to 0} = 0$$

DIFFERENTIATION

DEFINITION: A function f is a rule which assigns to every real number x a unique real number $f(x)$ or is undefined at x.

DEFINITION: The derivative of a function $f(x)$ is the function

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Example: Find the derivative of $f(x) = \sqrt{x}$ $(x \geq 0)$.

Sol^n

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

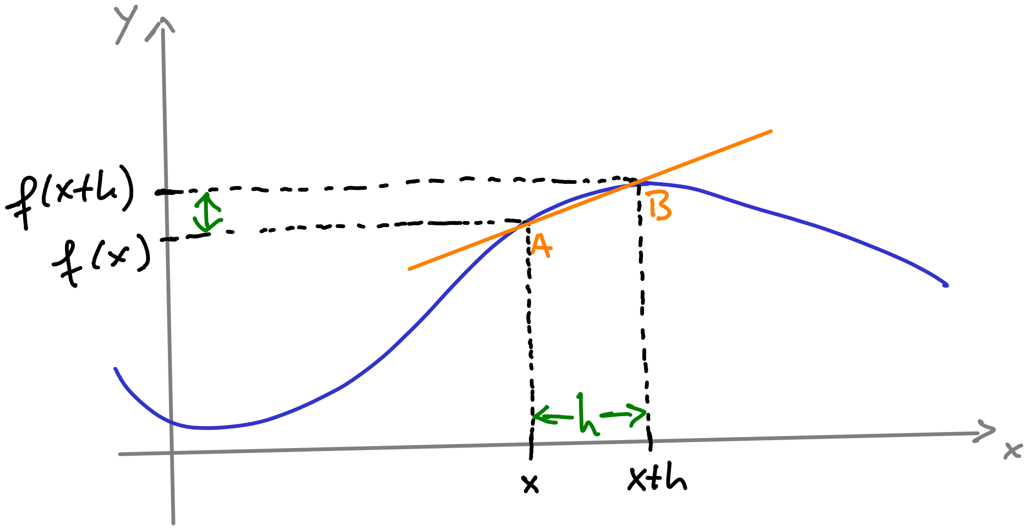
$(a-b)(a+b) = a^2 - b^2$

$$= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Interpretation of derivatives

Consider a function $f(x)$ and its graph.



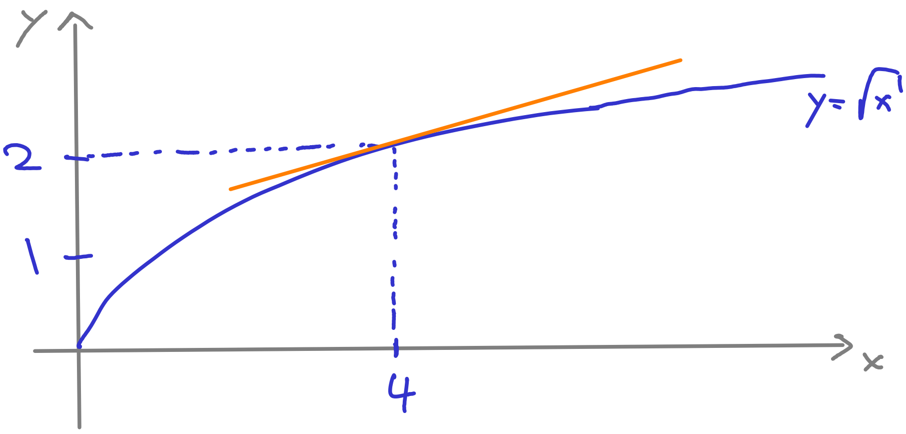
Slope of line AB $= \dfrac{f(x+h) - f(x)}{h}$

In the limit when $h \to 0$ we obtain the slope of the tangent line to the graph of f at the point $(x, f(x)) = A$.

In general: $f'(x)$ is the slope of the graph of f at $(x, f(x))$, if $f'(x)$ exists.

Example: Find the equation of the tangent to the graph of $f(x) = \sqrt{x}$ at $x=4$.

Sol^n:



A $x=4$ $f(x)=2$ and $f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{4}$

In general, a line is given by a function of the form

$$\boxed{\ell(x) = mx + b}$$

slope y-axis intercept

In the example, the tangent in question is given by $t(x) = \frac{1}{4}x + b$ which needs to contain the point $(4,2)$, i.e. $t(4)=2$ must hold.

This shows that $b=1$ and

$$t(x) = \frac{1}{4}x + 1$$

Example: Find the derivative of $f(x) = \sin(x)$.

Sol^n: $$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \left(\sin(x)\underbrace{\left(\frac{\cos(h) - 1}{h}\right)}_{\to 0} + \cos(x)\underbrace{\left(\frac{\sin(h)}{h}\right)}_{\to 1} \right)$$

$$= \cos(x)$$

Limits at infinity

Consider $f(x) = \dfrac{x}{\sqrt{x^2+1}}$; it's defined everywhere.

Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to -\infty} f(x)$

Sol^n: Since $$\frac{x}{\sqrt{x^2+1}} \cdot \frac{1/x}{1/x} = \frac{x/x}{\frac{\sqrt{x^2+1}}{x}} \overset{x\geq 0}{=} \frac{1}{\sqrt{\frac{x^2+1}{x^2}}}$$

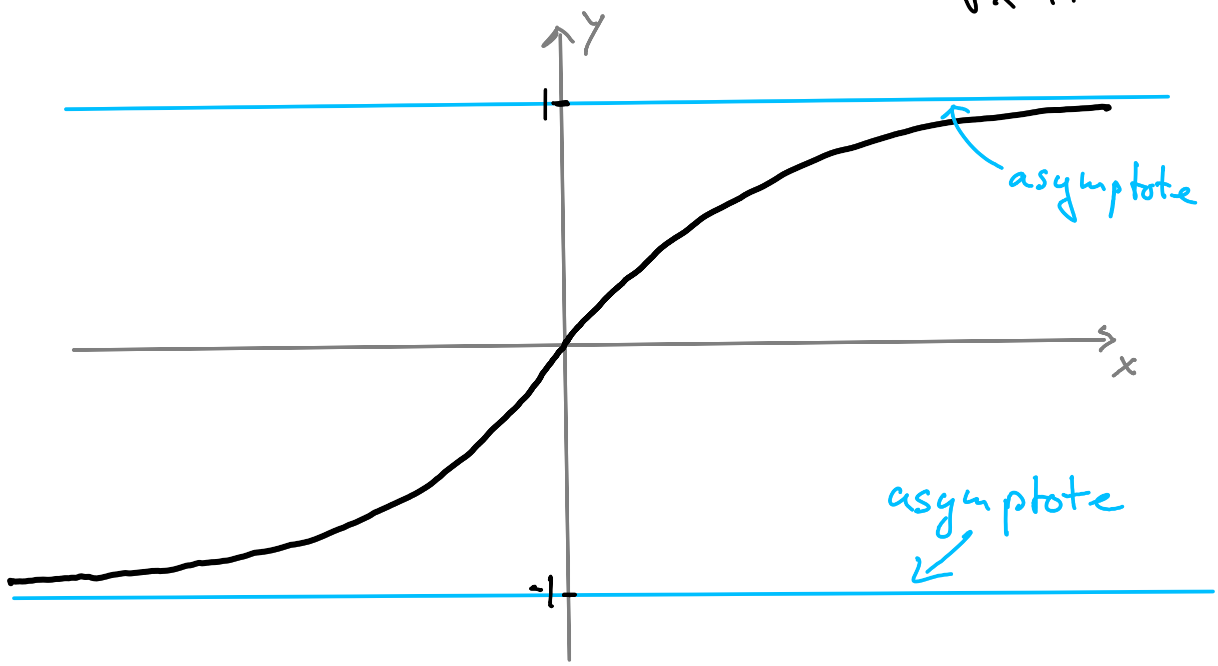
$$= \frac{1}{\sqrt{1+1/x^2}} \xrightarrow{x\to\infty} 1$$

So $$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = 1$$

For large negative x, we get $\dfrac{x}{\sqrt{x^2+1}} = \dfrac{-1}{\sqrt{1+1/x^2}}$.

So $$\lim_{x\to -\infty} \frac{x}{\sqrt{x^2+1}} = -1$$

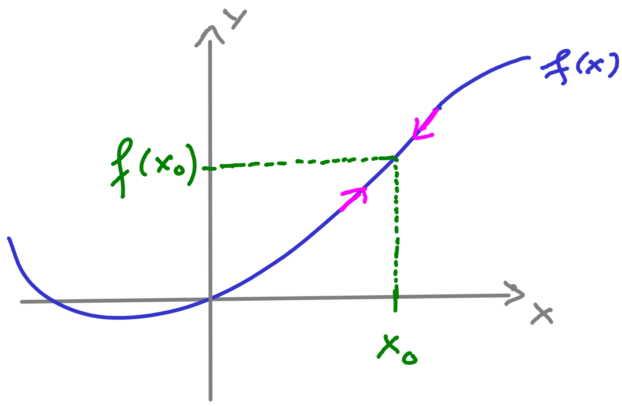
Let's sketch the graph of $f(x) = \frac{x}{\sqrt{x^2+1}}$



Continuity

A function $f(x)$ is continuous at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0)$$



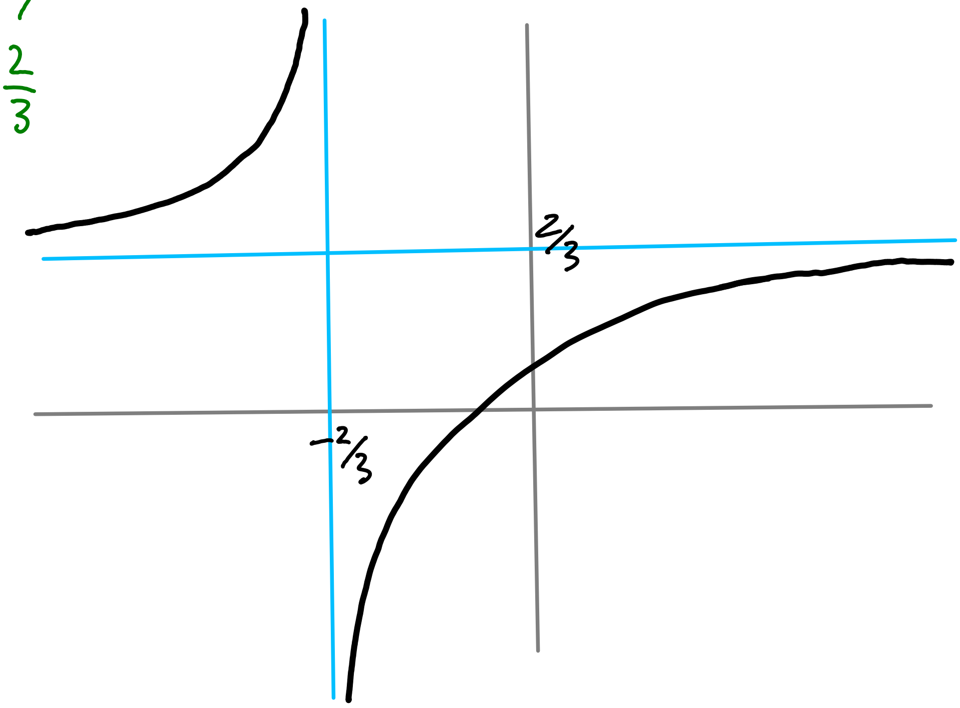
Example: What are the asymptotes and discontinuities of $f(x) = \frac{2x-5}{3x+2}$?

Solⁿ: We find $\lim_{x \to \infty} \frac{2x-5}{3x+2} = \frac{2}{3}$ because

$$\frac{2x-5}{3x+2} = \frac{2x-5}{3x+2} \cdot \frac{1/x}{1/x} = \frac{2 - 5/x}{3 + 2/x}$$

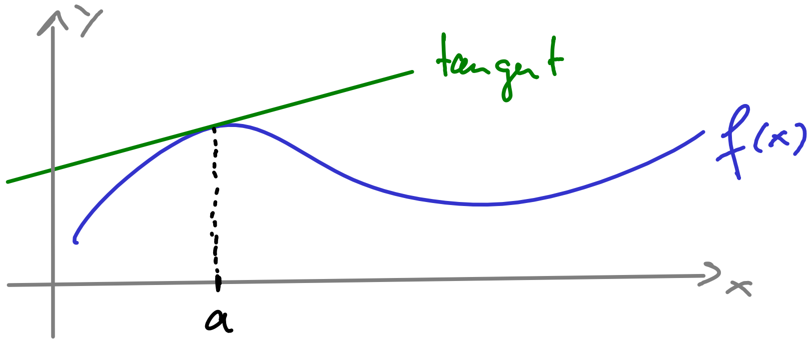
Similarly $\lim_{x \to -\infty} \frac{2x-5}{3x+2} = \frac{2}{3}$

The discontinuity is where the denominator becomes 0 which is $-\frac{2}{3}$



Tangent lines

Problem: Given a continuous function $f(x)$, find the equation for the tangent line to the graph of f at $(a, f(a))$.



Sol^n: The tangent line has slope $f'(a) = \frac{df}{dx}(a)$, so is given by

$$t(x) = f'(a)x + b$$

satisfying $t(a) = f(a)$. So $b = f(a) - f'(a)a$

Roots of functions

Definition: A root of a function $f(x)$ is a number a s.t. $f(a) = 0$.

In other words, an x-intercept of f.

Example: Find the roots of

$$f(x) = 6x^2 + 30x + 18$$

Sol^n: Solve $6x^2+30x+18=0$ for x.

Divide by 6: $x^2+5x+3=0$

$$x = -\frac{5}{2} \pm \sqrt{\frac{25}{4} - 3} = -\frac{5 \pm \sqrt{13}}{2}$$

Example: How many real roots, does

$x^3 - 2x^2 = 20$ have?

Rmk: A sum of multiples of powers of x is called a polynomial.

The highest power is the degree of the polynomial.

Theorem: A polynomial of degree d has at most d complex roots.

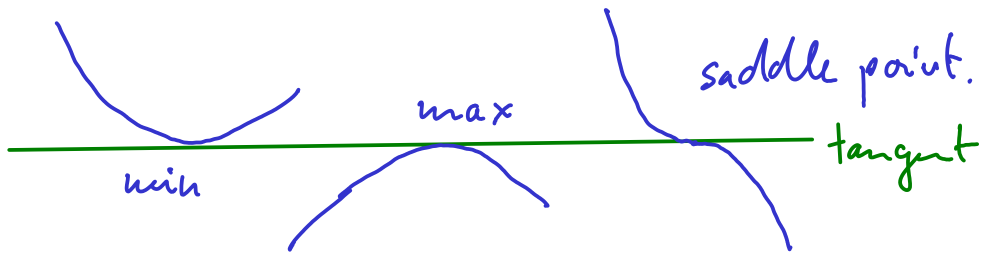
Back to the example: Let write $f(x) = x^3 - 2x^2 - 20$,

so we are looking for the roots of f.

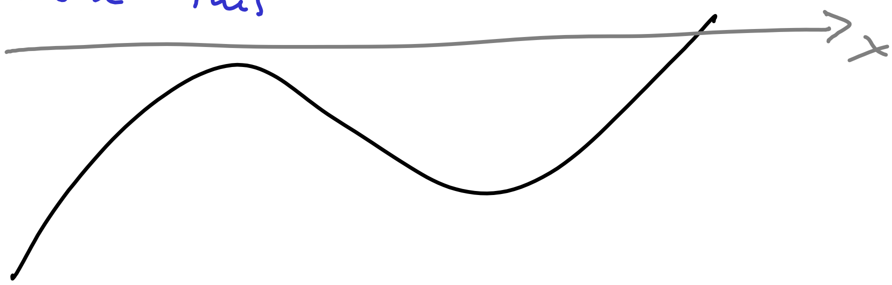
$$\left.\begin{array}{ll} \text{For } x \to \infty & f(x) \to \infty \\ \text{For } x \to -\infty & f(x) \to -\infty \end{array}\right\} \text{ shows that there is at least one zero / root}$$

$$f'(x) = 3x^2 - 4x = (3x-4)x$$

The zeros/roots of $f'(x)$ correspond to local maxima or minima or saddle points, because slope (of tangent line) equal to zero means horizontal tangent



Now we know that the graph of $f(x) = x^3 - 2x^2 - 20$ looks like this



The min and max of f are at 0 and $4/3$ (the zeros of f')

Now $f(0) = -20$, $f(4/3) = \frac{64}{27} - \frac{32}{9} - 20 < 0$

So, the x-axis is above the local max at $x=0$ and so $f(x)$ has precisely one real root.

A bit more on assymptotes and discontinuities

An assymptote for a graph of a function is a vertical or horizontal line which the graph approaches, gets arbitrarily close but never touches.

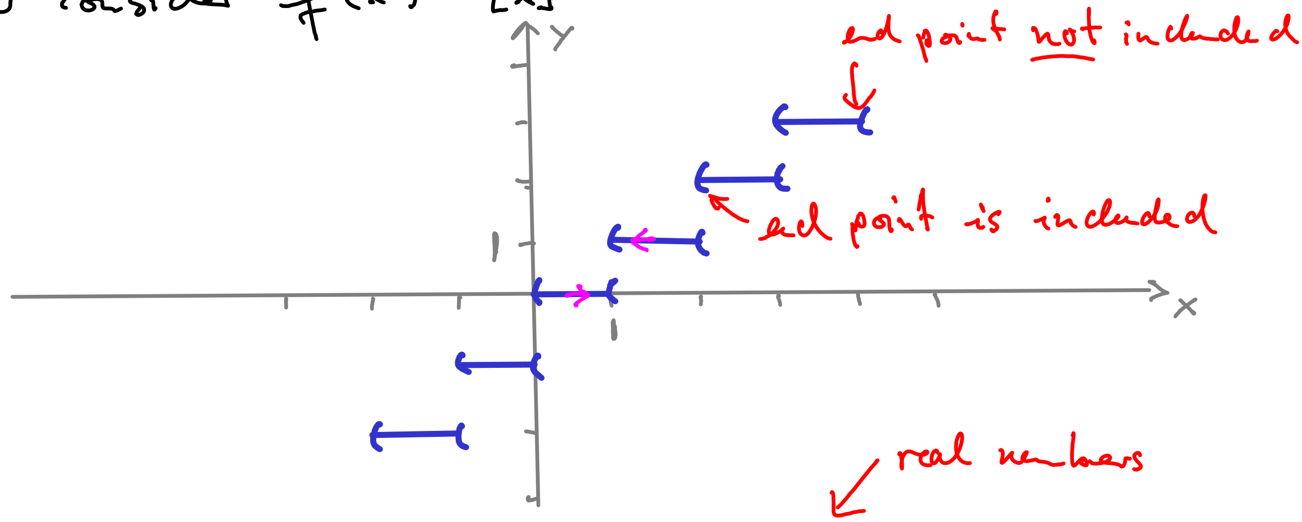
A discontinuity of a function f is a "gap", i.e. a value x_0 s.t. $f(x_0)$ is not defined or

$$\lim_{x \to x_0} f(x) \text{ does not exist.}$$

<u>Examples:</u> For a real number x, let $\lfloor x \rfloor$ be defined as the largest integer less or equal to x.

$\lfloor 2.781 \rfloor = 2$; $\lfloor -5.6 \rfloor = -6$; $\lfloor 123.56891 \rfloor = 123$

Now consider $f(x) = \lfloor x \rfloor$



real numbers

This function is defined for all $x \in \mathbb{R}$ and has discontinuities at $x \in \mathbb{Z}$ integers

<u>NOTATION:</u> $\lim_{x \to 1^-} \lfloor x \rfloor = 0$ and $\lim_{x \to 1^+} \lfloor x \rfloor = 1$

from left from right

Example: Consider $f(x) = \dfrac{2x^2}{(x-1)(x+2)}$.

Sketch the graph of f.

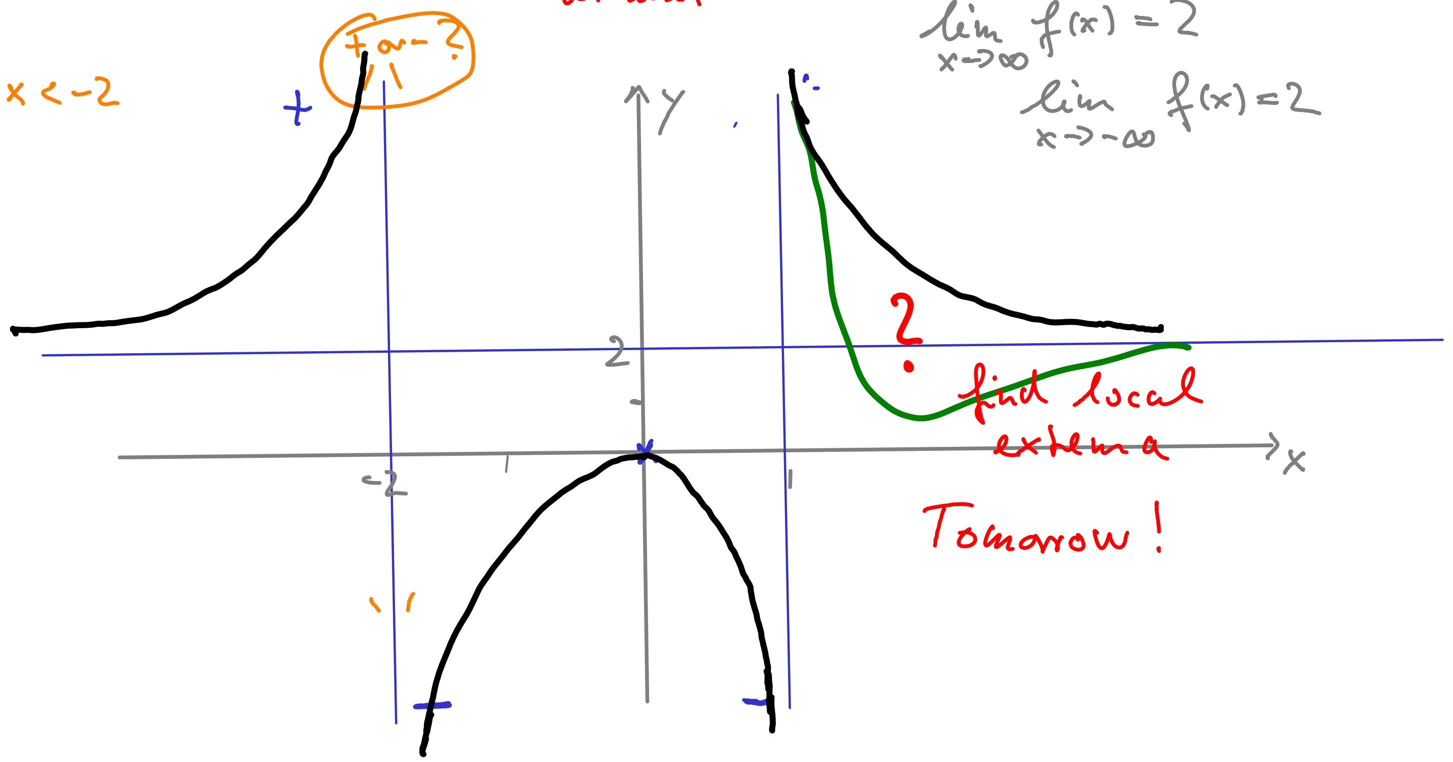
Domain $(f) = \mathbb{R} \setminus \{1,-2\}$ ← without

x-intercept: solve $f(x)=0$

So $x=0$ is x-intercept

$\lim_{x\to\infty} f(x) = 2$

$\lim_{x\to-\infty} f(x) = 2$



Maxima, Minima and Saddle Points

So let's find the real shape of the graph of

$$f(x) = \frac{2x^2}{(x-1)(x+2)}$$

A critical point of a function is a zero/root of its derivative.

Calculating derivatives

① Some derivatives you should know:

$f(x)$	$f'(x) = \frac{df}{dx}(x)$
x^n	$n x^{n-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$

② Rules to calculate more difficult derivatives

Sums: $\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{df}{dx}(x) + \frac{dg}{dx}(x)$

Constant factor: $\frac{d}{dx}\left(K f(x)\right) = K \frac{df}{dx}(x)$, $K = \text{const.}$

Product Rule: $\frac{d}{dx}\left(f(x)\,g(x)\right) = \frac{df}{dx}(x)\,g(x) + f(x)\,\frac{dg}{dx}(x)$

Chain Rule: $\frac{d}{dx}\left(f(g(x))\right) = \frac{df}{dx}(g(x))\,\frac{dg}{dx}(x)$

Quotient Rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{df}{dx}(x)\,g(x) - f(x)\,\frac{dg}{dx}(x)}{[g(x)]^2}$

Examples: * $\frac{d}{dx}\left(\sin^2(x)\right) = ?$

let $f(x) = x^2$ and $g(x) = \sin(x)$,

then $f(g(x)) = \sin^2(x)$ and by the chain rule

$$\frac{d}{dx}\left(\sin^2(x)\right) = 2\sin(x)\cos(x)$$

* $\frac{d}{dx}\left(\ln(3x^2+4x-2)\right)$. With $f(x) = \ln(x)$

and $g(x) = 3x^2+4x-2$, we get that

$f(g(x)) = \ln(3x^2+4x-2)$ and by the chain rule

$$\frac{d}{dx}\left(\ln(3x^2+4x-2)\right) = \frac{1}{3x^2+4x-2}(6x+4)$$

* $\frac{d}{dx}\left(\cos(e^{x^2})\right) = -\sin\left(e^{x^2}\right)e^{x^2}\,2x$

Now back to $f(x) = \frac{2x^2}{(x-1)(x+2)}$

Note: $(x-1)(x+2) = x^2+x-2$

Using the quotient rule $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$,

we get

$$f'(x) = \frac{4x\,(x^2+x-2) - 2x^2(2x+1)}{(x-1)^2\,(x+2)^2}.$$

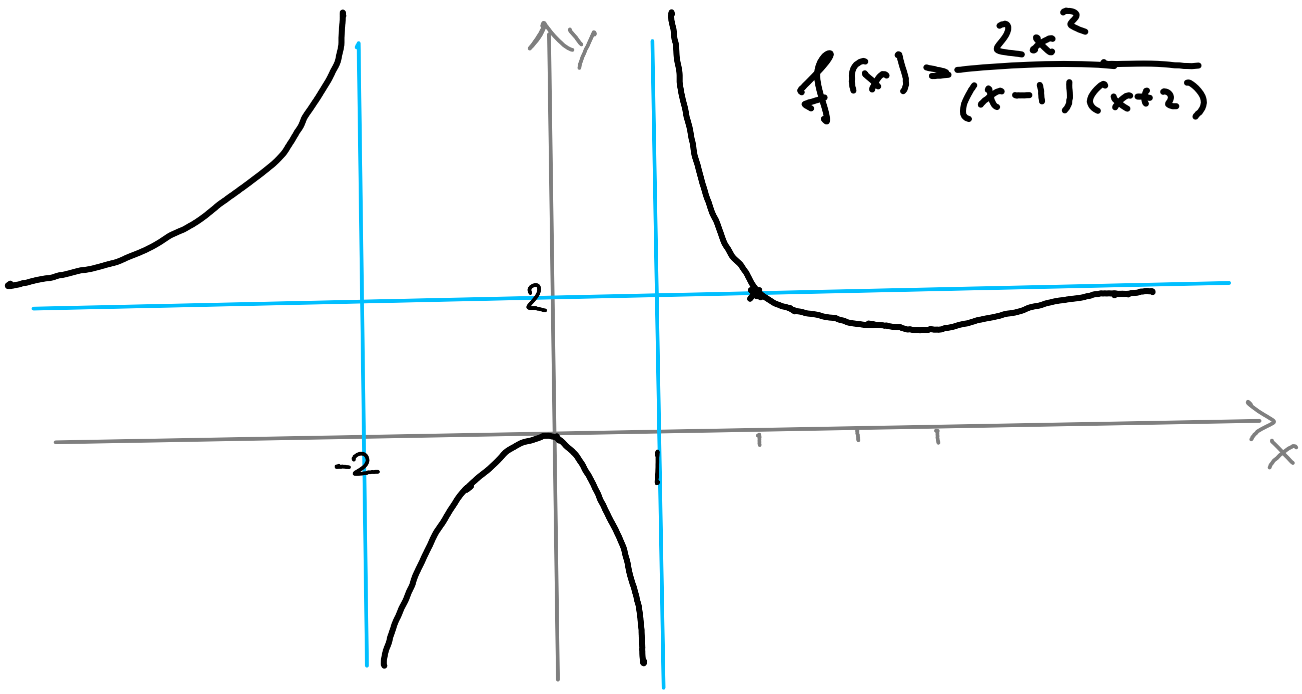
The critical points of f are the zeros of f'

So solve $4x(x^2+x-2) - 2x^2(2x+1) = 0$

$$= \underline{4x^3} + 4x^2 - 8x - \underline{4x^3} - 2x^2$$

$$= 2x^2 - 8x = x(2x-8)$$

So $x = 0$ or $2x - 8 = 0$, i.e. $x = 4$.



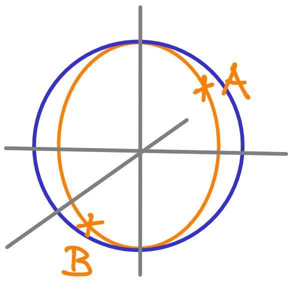
Intermediate Value Theorem:

Suppose that f is continuous at all points x with $a \leq x \leq b$. Assume also that $f(a)f(b) \leq 0$. Then there is a c with $a \leq c \leq b$ such that $f(c)=0$.

We already used it when we showed that $x^3+1=3x$ has three real solutions.

Another example:

Take any great circle on the earth.



Fact: There are diametrically opposite points A and B at which the air pressure (or temperature) is equal.

The idea is that the function

$$f(\theta) = \text{pressure at } A - \text{pressure at } B$$

is continuous. Here θ varies from 0 to π.

Also note that $f(0) = -f(\pi)$

So either $f(0) = f(\pi) = 0$ and we found the two points, or $f(0)$ and $f(\pi)$ have opposite signs and by the theorem $f(\theta)=0$ for some θ in between 0 and π.

Applications of derivatives

① We already calculated equations for tangent lines.

② Derivatives can be used to calculate limits.

Remember $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(h)}{h}$.

So for very small h

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

or

$$h f'(x) \approx f(x+h) - f(x)$$

or

$$f(x) = f(x+h) - h f'(x)$$

Suppose $f(x)$ and $g(x)$ are continuous functions and that for some c we have $f(c) = g(c) = 0$.

Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} \approx \lim_{x \to c} \frac{f(x+h) - h f'(x)}{g(x+h) - h g'(x)}$$

and for x close to c $f(x+h) \approx 0$ and $g(x+h) \approx 0$.

So $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} \approx \lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)}$

L'Hospital' Rule

If $f(x)$ and $g(x)$ are continuous and $f(c) = g(c) = 0$,

then $\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$.

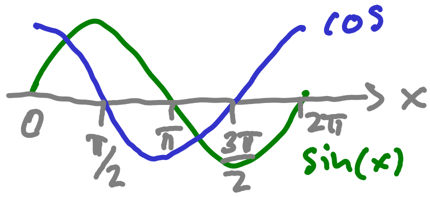
Example: Evaluate $\lim_{x \to \pi/2} \tan(x) - \sec(x)$.

Recall: $\tan(x) = \frac{\sin(x)}{\cos(x)}$

$$\sec(x) = \frac{1}{\cos(x)}$$

So $\lim_{x \to \pi/2} \tan(x) - \sec(x) = \lim_{x \to \pi/2} \frac{\sin(x) - 1}{\cos(x)}$

Recall: $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$



So we use L'Hospital's Rule.

$$\lim_{x \to \pi/2} \frac{\sin(x) - 1}{\cos(x)} = \lim_{x \to \pi/2} \frac{\cos(x)}{-\sin(x)} = \frac{0}{-1} = 0$$

Example: Find $\lim_{x \to 0} \frac{\sin(x)\cos(x) - x}{x^3}$.

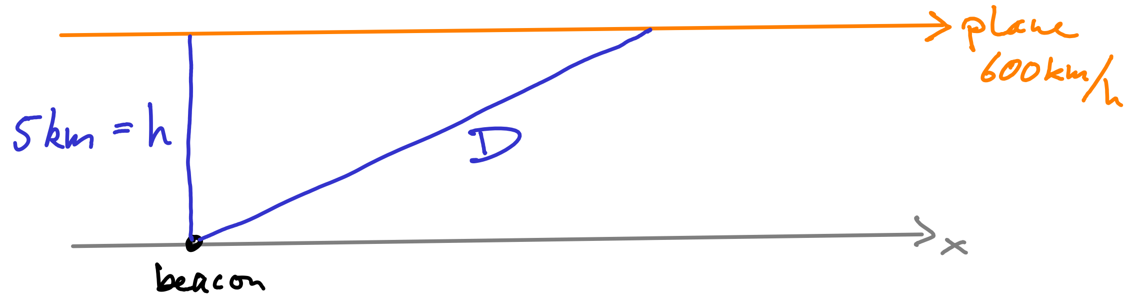
Use L'Hospital $\lim_{x \to 0} \frac{\sin(x)\cos(x) - x}{x^3} = \lim_{x \to 0} \frac{\cos^2(x) - \sin^2(x) - 1}{3x^2}$

$$= \lim_{x \to 0} \frac{-4\cos(x)\sin(x)}{6x} = \lim_{x \to 0} \frac{4\sin^2(x) - 4\cos^2(x)}{6} = -\frac{2}{3}$$

More applications

The derivative can be thought of as "rate of change".

Problem: An aircraft is flying horizontally at a speed of 600 km/h. How fast is the distance between a radio beacon and the aircraft increasing 1 minute after the plane passes 5 km above the beacon?



Sol^n: We need to find $\frac{dD}{dt}$ at $t=1$.

When $t=1$, then $x=10$ and

$$D^2 = 5^2 + x^2$$

Note: $D = D(x)$ but $x = x(t)$.

By the chain rule:

$$\frac{dD}{dt} = \frac{dD}{dx}\frac{dx}{dt}$$

So $D = \sqrt{5^2+x^2} = (25+x^2)^{1/2}$

and
$$\frac{dD}{dt} = \frac{1}{2}(25+x^2)^{-1/2}\, 2x \frac{dx}{dt}$$

horizontal speed

$$= 10(25+x^2)^{-1/2}x$$

Calculate in minutes!

$$= \frac{10x}{(25+x^2)^{1/2}}$$

So when $t=1$, $x=10$ and

$$\frac{dD}{dt}(1\,\text{min}) = \frac{10\times 10}{(125)^{1/2}} = \frac{20}{\sqrt{5}} \qquad \left[\frac{\text{km/min}\ \text{km}}{\text{km}} = \text{km/min}\right]$$

Problem: How fast does the volume of a balloon increase, when its radius increases at a constant rate of 2cm per minute?

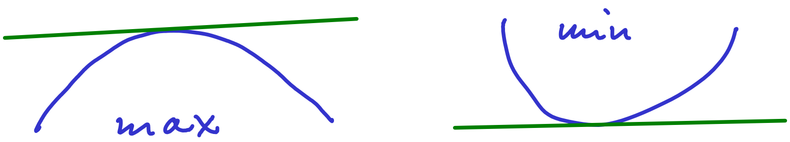
Sol^n: The volume V is given by

$$V = V(r) = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt} = 8\pi r^2$$

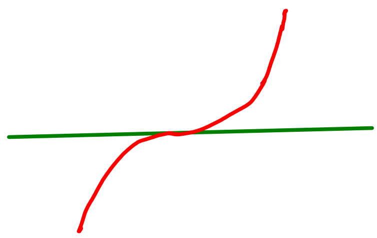
This depends on the current radius!

Derivatives can be used to find where the function increases, decreases, has a maximum or minimum, has a point of inflection.

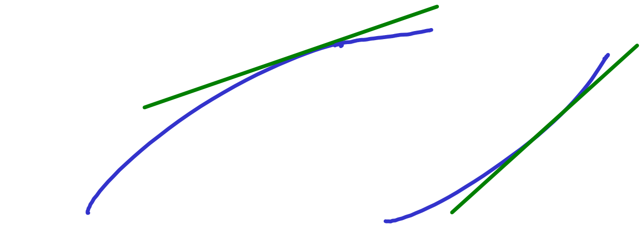


horizontal tangent, i.e. slope = 0

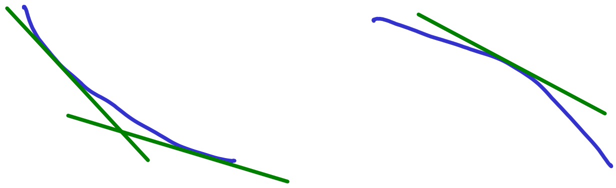
Warning:



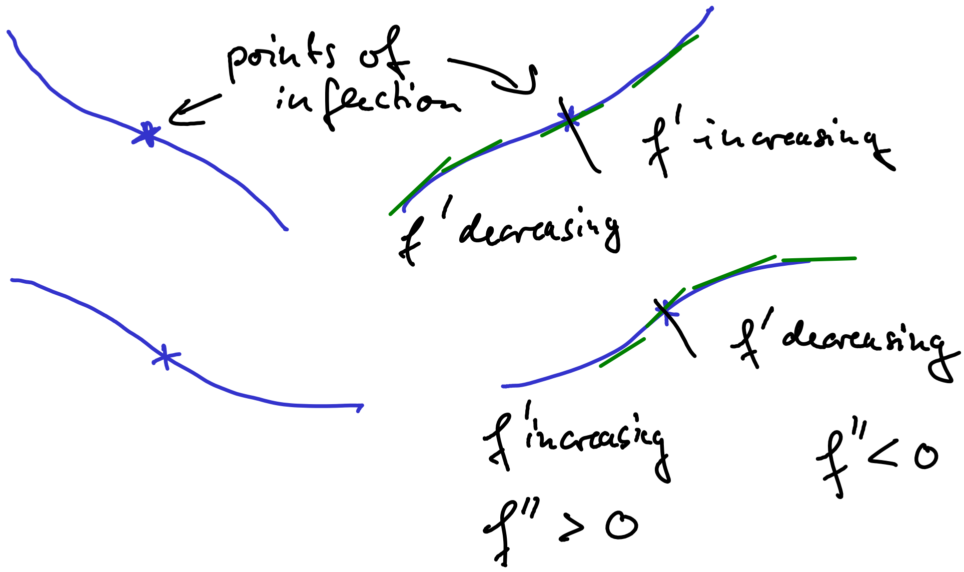
also has horizontal tangent.



f increasing is equivalent to positive slope; $f' > 0$



f decreasing is equivalent to negative slope; $f' < 0$



In particular, at a max of f, f' is decreasing and $f'' > 0$. At a min of f, f' is increasing and $f'' < 0$.

Optimisation Problems

Problem: You have the choice of a rectangular shaped garden of 60 square metres. What side lengths should the garden have in order to minimise the length of the fence around it?

Sol^n: Let a and b be the side lengths.

Then area $= ab = 60$ and perimeter is

$$P = 2a + 2b = \frac{120}{b} + 2b = \frac{2b^2+120}{b}.$$

By the quotient rule $\left[\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \right]$

$$\frac{dP}{db} = \frac{4bb - (2b^2+120)}{b^2} = \frac{2b^2-120}{b^2}.$$

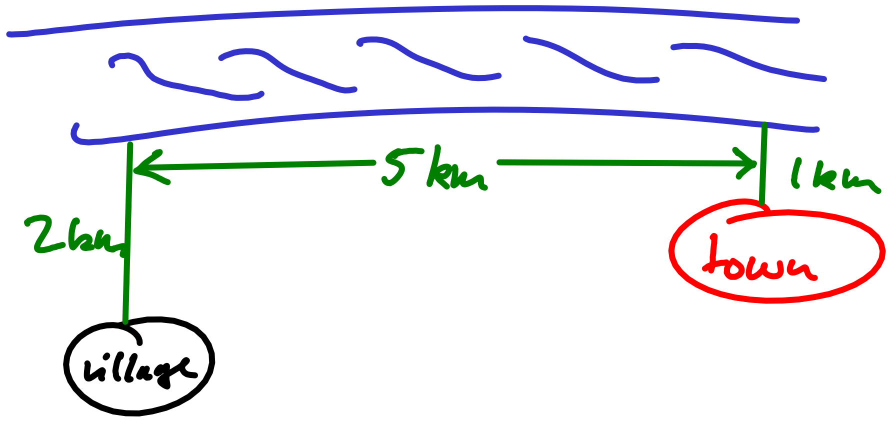
Now $\frac{dP}{db} = 0$ if and only if $2b^2 - 120 = 0$

or $b^2 = 60$, i.e. $b = \pm\sqrt{60}$.

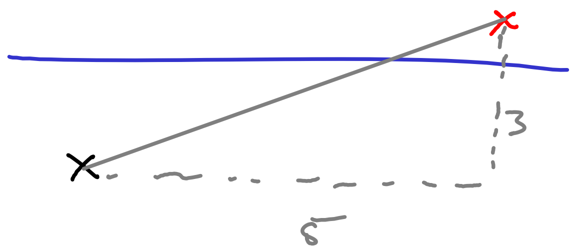
Since $-\sqrt{60}$ makes no sense, the garden should be a square of side length $\sqrt{60}$.

Problem: A farmer needs to go into town but needs to wash his potatoes in the river on the way. Both, his village is 2km and the town 1km away from the river shore. The town and village are 5km appart in the direction of the river.

Find the shortest path.



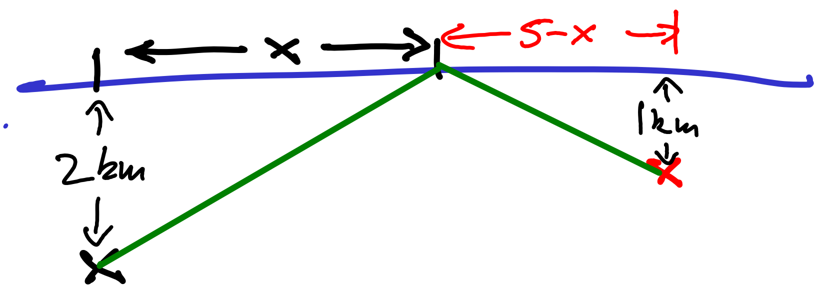
Sol^n



graphical

$\sqrt{34}$

Using functions: let $f(x)$ be the length of the path, when you meet the river at x



$$f(x) = \sqrt{4+x^2} + \sqrt{1+(5-x)^2}$$

$$\frac{df}{dx} = \frac{x}{\sqrt{4+x^2}} + \frac{-(5-x)}{\sqrt{1+(5-x)^2}}$$

$$= \frac{x\sqrt{1+(5-x)^2} + (x-5)\sqrt{4+x^2}}{\sqrt{4+x^2}\,\sqrt{1+(5-x)^2}}$$

$$\frac{df}{dx} = 0 \quad \text{when} \quad x\sqrt{1+(5-x)^2} + (x-5)\sqrt{4+x^2} = 0$$

or $$x\sqrt{1+(5-x)^2} = (5-x)\sqrt{4+x^2}$$

or $$x^2\left(1+(5-x)^2\right) = (5-x)^2\,(4+x^2)$$

or $$x^2 + \cancel{x^2(5-x)^2} = \cancel{x^2(5-x)^2} + 4(5-x)^2$$

or $$0 = 4(25-10x+x^2) - x^2$$

$$= 3x^2 - 40x + 100$$

or $$0 = x^2 - \frac{40}{3}x + \frac{100}{3}$$

$$x = \frac{20}{3} \pm \sqrt{\frac{400}{9} - \frac{300}{9}}$$

$$= \frac{20}{3} \pm \frac{10}{3} = \begin{cases} 10 \\ \frac{10}{3} \end{cases}$$

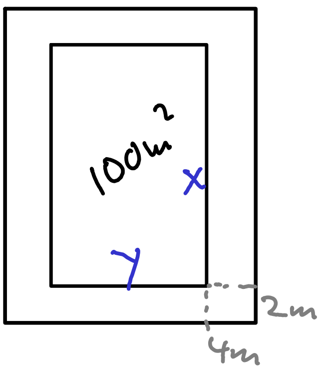
The problem suggests $x = \frac{10}{3}$.

Then $$f\left(\tfrac{10}{3}\right) = \sqrt{4+\frac{100}{9}} + \sqrt{1+\left(5-\frac{10}{3}\right)^2}$$

$$= \sqrt{\frac{136}{9}} + \sqrt{\frac{34}{9}} = \sqrt{4\cdot\frac{34}{9}} + \sqrt{\frac{34}{9}}$$

$$= \sqrt{34}$$

<u>Problem</u>: A billboard is to have 100 m^2 of printed area with 2 m margins at top and bottom and 4 m margins on the sides.



Find the dimensions if the total area is to be minimal.

$xy = 100$

total area $A = (x+4)(y+8)$

$= xy + 4y + 8x + 32$

$= 132 + 4y + 8x$

Also $y = \frac{100}{x}$. So $A(x) = 132 + \frac{400}{x} + 8x$

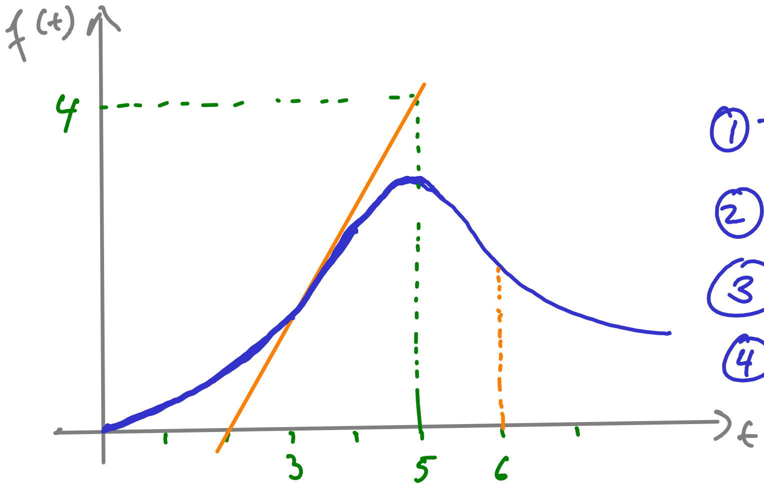
$\frac{dA}{dx} = 8 - \frac{400}{x^2}$ and $\frac{dA}{dx} = 0 = 8 - \frac{400}{x^2}$

means $8x^2 = 400$ or $x^2 = 50$ or $x = \pm\sqrt{50}$

<u>So the dimensions are</u> $\sqrt{50}$ by $2\sqrt{50}$.

Understanding / plotting curves

<u>Example</u>: A car is driving on a long straight road. At time t, its distance from the start is $f(t)$ and the graph of f is



Determine when

1. The speed is positive
2. The speed is negative
3. Car is accelerating
4. The fastest speed the car reaches.

$f'(t)$ + + + ... + + 0 - - - - - -

Speed is derivative of distance.

Acceleration is rate of change of speed, i.e second derivative of f

Asw: 1. positive speed $0 \le t \le 5$

2. negative speed $5 \le t$

3. accelerating $0 \le t \le 3$

4. fastest speed = slope of tangent at point of inflection, which is $4/3$

Problem: Sketch the graph of $y = x e^{-x^2/2} = f(x)$

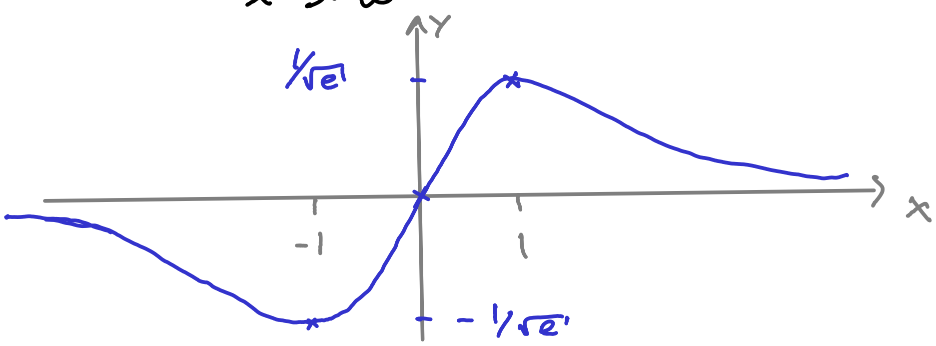
x-axis intercept: $y = 0 \iff x = 0$

y-axis intercept: $x = 0$, so $(0, 0)$

$$f'(x) = e^{-x^2/2} + x e^{-x^2/2}(-x) = (1 - x^2) e^{-x^2/2}$$

critical points: $f'(x) = 0 \iff 1 - x^2 = 0$, or $x = \pm 1$

$$\lim_{x \to \infty} f(x) = 0, \quad \lim_{x \to -\infty} f(x) = 0$$



Logarithms and exponents

Laws of exponents

* $(a^b)^c = a^{bc}$

If $b \in \mathbb{N}$, then $a^b = \underbrace{a\,a \cdots a}_{b \text{ many } a\text{'s}}$.

* $a^b a^c = a^{b+c}$

* If $a > 0$, then $a^b > 0$ for all b.

* $a^0 = 1$

* If $n \in \mathbb{N}$, then $a^{1/n} = \sqrt[n]{a}$.

* $a^{-b} = \dfrac{1}{a^b}$

Logarithms are the inverses of exponentiation.

$2^4 = 16$ $\qquad$ $\log_2(16) = 4$

$4^3 = 64$ $\qquad$ $\log_4(64) = 3$

$4^{1/2} = 2$ $\qquad$ $\log_4(2) = \frac{1}{2}$

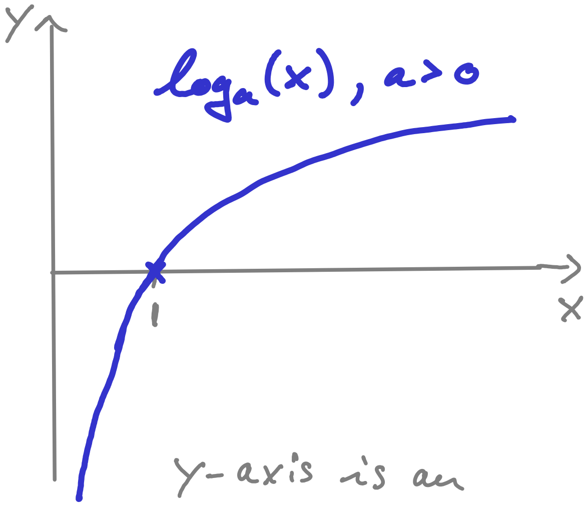
$8 = 2^3 = (4^{1/2})^3 = 4^{3/2}$ $\qquad$ $\log_4(8) = \frac{3}{2}$

Rules for logarithms

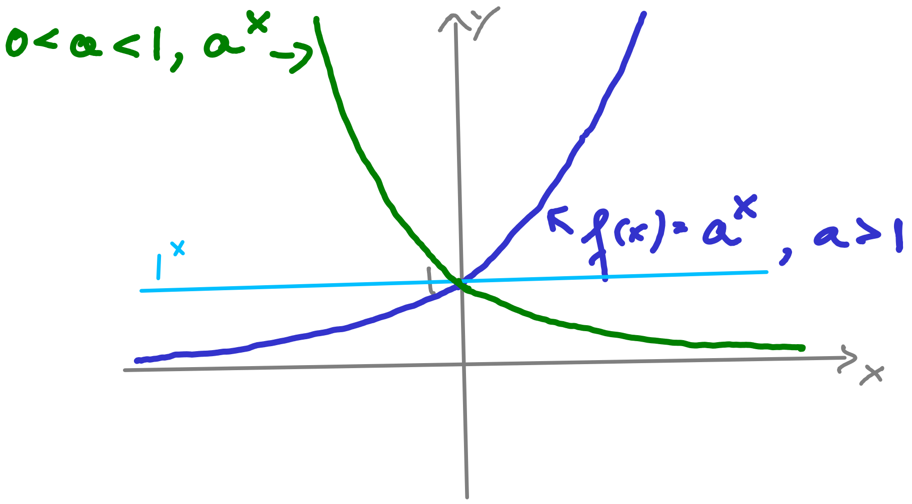
* $\log_a(bc) = \log_a(b) + \log_a(c)$

* $\log_a(b^c) = c \log_a(b)$

* $\log_a(1) = 0$
* For $a > 0$, $\log_a : \mathbb{R}_{>0} \longrightarrow \mathbb{R}$
* $a^{\log_a(b)} = b$
* $\lim_{x \to \infty} \log_a(x) = \infty$



y-axis is an assymptote



$\lim_{x \to -\infty} a^x = 0$, $a>1$

Recall: Our favourite base for exponentiation and logarithms is e (Euler's number)

This is because $\frac{d}{dx}(e^x) = e^x$

Now what is $\frac{d}{dx}(a^x)$?

$$a^x = \left(e^{\log_e(a)}\right)^x = e^{x \log_e(a)}$$

So $\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{x \log_e(a)}\right) = \log_e(a)\, e^{x \log_e a} = \log_e(x)\, a^x$

Convention: Instead of $\log_e$ we write ln or log.

One can use logarithm to solve equations where the target occurs in the exponent

Example: Solve $2^{x^2-2x+1} = 64$.

Sol^n: Apply $\log_2$ on both sides to get

$$\log_2\left(2^{x^2-2x+1}\right) = \log_2(64)$$

or $\quad x^2-2x+1 = 6$

or $\quad x^2-2x-5 = 0$

So $\quad x = 1 \pm \sqrt{6}$

A proper definition of $\ln(x)$ involves an integral.

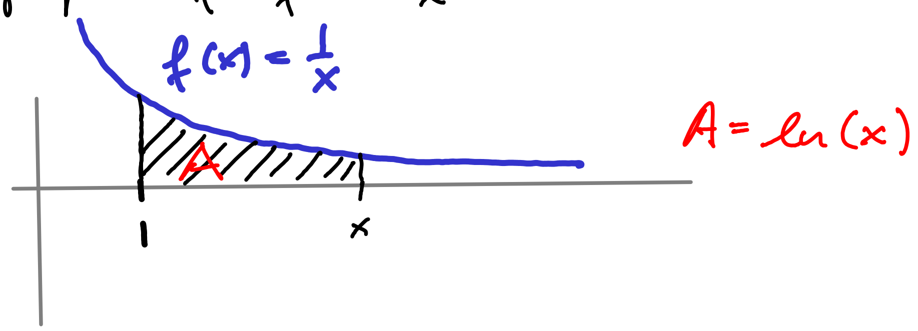
Recall that $\frac{d}{dx}(x^n) = n\,x^{n-1}$, $n \in \mathbb{R}$

In particular, for $n \in \mathbb{Z}$, we get x^{n-1} as the derivative of $\frac{1}{n}x^n$. Except for $n=0$!

So x^{-1} does never occur as derivative here.

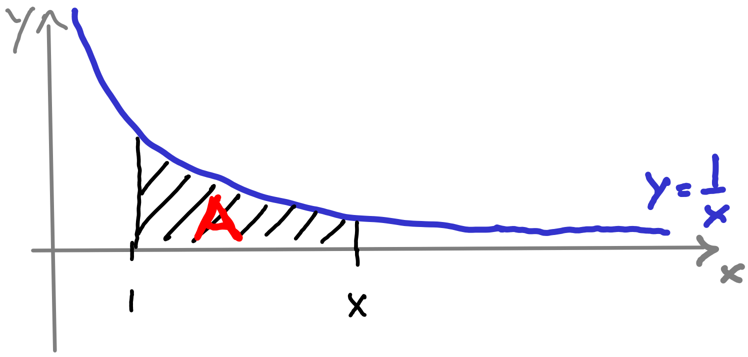
Fact: $\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x} = x^{-1}$

So the proper definition of $\ln(x)$ is as the area under the graph of $f(x) = \frac{1}{x}$ and above the x-axis.



Logarithm defined properly

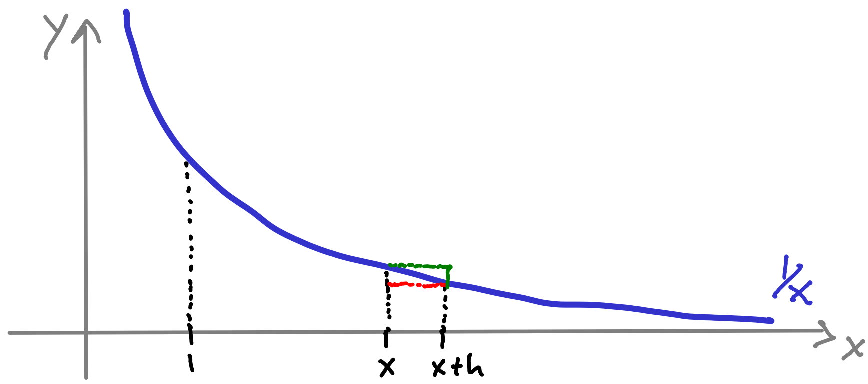
Definition: For $x > 0$ let A be the are shown below.



Define $\quad \ln(x) = \begin{cases} A & x \geq 1 \\ -A & x < 0 \end{cases}$

Theorem: For $x > 0$, we have $\dfrac{d}{dx}\left(\ln(x)\right) = \dfrac{1}{x}$.

Proof:



$$\frac{d}{dx}\left(\ln(x)\right) = \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h}$$

$$\frac{h}{x+h} < \ln(x+h) - \ln(x) < \frac{h}{x}$$

Dividing by h gives

$$\frac{1}{x+h} < \frac{\ln(x+h) - \ln(x)}{h} < \frac{1}{x}$$

So the limit is squeezed between $\lim_{h \to 0} \frac{1}{x+h} = \frac{1}{x}$ and $\frac{1}{x}$

and so $\dfrac{d}{dx}\left(\ln(x)\right) = \dfrac{1}{x}$.

Consequences:

① For $x, y > 0$, then $\ln(xy) = \ln(x) + \ln(y)$.

The reason is $\frac{d}{dx}(\ln(xy)) = \frac{y}{xy} = \frac{1}{x} = \frac{d}{dx}(\ln(x))$

So $\frac{d}{dx}(\ln(xy) - \ln(x)) = 0$ and hence

$\ln(xy) - \ln(x) = c$, a constant.

With $x = 1$, we get $\ln(y) = c$, which then

gives $\ln(xy) - \ln(x) = \ln(y)$, or $\ln(xy) = \ln(x) + \ln(y)$.

② One can also show $\ln(x^k) = k\ln(x)$

Application: Find the derivative of

$$f(x) = \frac{(x+1)^2(x-2)^3(x+4)}{(x-1)^2(x-6)}$$

Idea: Use $\ln$ to turn the product into a sum.

$$\ln(f(x)) = \ln\left((x+1)^2(x-2)^3(x+4)(x-1)^{-2}(x-6)^{-1}\right)$$

$$= \ln((x+1)^2) + \ln((x-2)^3) + \ln(x+4) + \ln((x-1)^{-2}) + \ln((x-6)^{-1})$$

$$= 2\ln(x+1) + 3\ln(x-2) + \ln(x+4) - 2\ln(x-1) - \ln(x-6)$$

Next take derivative

$$\frac{f'(x)}{f(x)} = \frac{2}{x+1} + \frac{3}{x-2} + \frac{1}{x+4} - \frac{2}{x-1} - \frac{1}{x-6}$$

So $f'(x) = f(x)\left[\frac{2}{x+1} + \frac{3}{x-2} + \frac{1}{x+4} - \frac{2}{x-1} - \frac{1}{x-6}\right]$.

$$= \frac{(x+1)^2(x-2)^3(x+4)}{(x-1)^2(x-6)}\left[\frac{2}{x+1} + \frac{3}{x-2} + \frac{1}{x+4} - \frac{2}{x-1} - \frac{1}{x-6}\right]$$

A little bit on differential equations

A differential equation, is an equation involving a variable and its derivative.

The simplest example is

$$\frac{dy}{dx} = ky \qquad , \quad k \text{ a const. and } y = y(x).$$

Are there any solutions? YES: $y = e^{kx}$

More general: $y = Ce^{kx}$, C a const.

This is a solution, because

$$\frac{dy}{dx} = Cke^{kx} = ky.$$

Are these all the solutions?

Suppose $y(x)$ and $z(x)$ are both solutions.

Then $\frac{d}{dx}\left(\frac{y}{z}\right) = \frac{y'z - z'y}{z^2} = \frac{kyz - kzy}{z^2} = 0$,

So $\frac{y}{z} = K = \text{const.}$, i.e $y = Kz$, that is any two solutions are multiples of each other.

Example: A cup of coffee in room at 20°C cools from 80°C to 50°C in five minutes. How long will it take to cool down to 40°C?

Newton: A hot object cools at a rate proportional to the difference between its temperature and the room temperature.

Let $T(t)$ be the temperature of the object at time t.

Newton: $\frac{dT}{dt} = k\left(T(t) - T_0\right)$, where T_0 is the room temperature.

We also know $T(0) = 80$ and $T(5) = 50$.

Question is: For which t is $T(t) = 40$?

$\frac{dT}{dt} = k(T - T_0)$. Let's put $S = T - T_0$.

Then $\frac{dS}{dt} = \frac{dT}{dt}$ and

$\frac{dS}{dt} = kS$ whose solution is $S = Ae^{kt}$.

So $T = S + T_0 = \boxed{Ae^{kt} + T_0 = T(t)}$ $\quad T_0 = 20$

Now use $T(0) = A + T_0 = 80$ or $A = 80 - T_0 = 60$,

and then $T(5) = 60e^{5k} + 20 = 50$

or $e^{5k} = \frac{1}{2}$ or $5k = \ln\left(\frac{1}{2}\right)$.

So $k = \frac{-\ln(2)}{5}$. To sum up: $T(t) = 60e^{-\frac{\ln(2)}{5}t} + 20$.

Finally: Solve $40 = 60 e^{-\frac{\ln(2)}{5} t} + 20$

$$\text{or} \quad \frac{1}{3} = e^{-\frac{\ln(2)}{5} t}$$

$$\text{or} \quad -\ln(3) = -\frac{\ln(2)}{5} t$$

$$\text{and so} \quad t = \frac{5\ln(3)}{\ln(2)} \approx 7.925.$$

Another example: A radioactive material has a half life of 1200 years. Given a sample of the material, what percentage remains after 10 years?

Fact: The quantity of radioactive material decays at a rate proportional to the amount of radioactive material.

Half life, is the time it takes for half the material to decay.

Let $y(t)$ be the amount of radioactive material at time t.

We know that $y(1200) = \frac{1}{2} y(0)$. Also $\frac{dy}{dt} = ky$,

so $y(t) = Ae^{kt}$, $y(0) = A$, $y(1200) = \frac{1}{2}A = Ae^{1200k}$,

which gives us $-\ln(2) = 1200k$, or $k = \frac{-\ln(2)}{1200}$.

Next: $y(10) = A e^{-\frac{\ln(2)}{1200} 10} = A e^{-\frac{\ln(2)}{120}} = A\left(\frac{1}{2}\right)^{\frac{1}{120}}$

$$= A * 0.9942$$

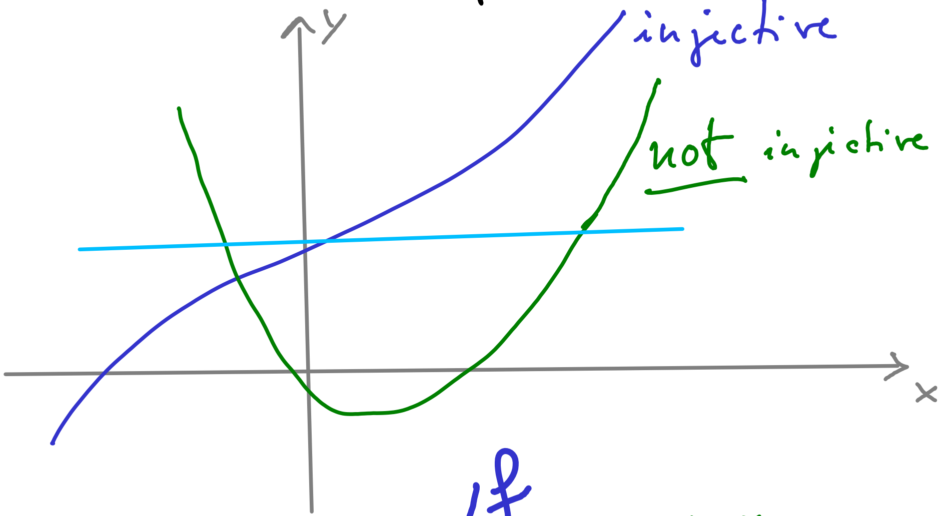
This says that after 10 years 99.42% of the material remain.

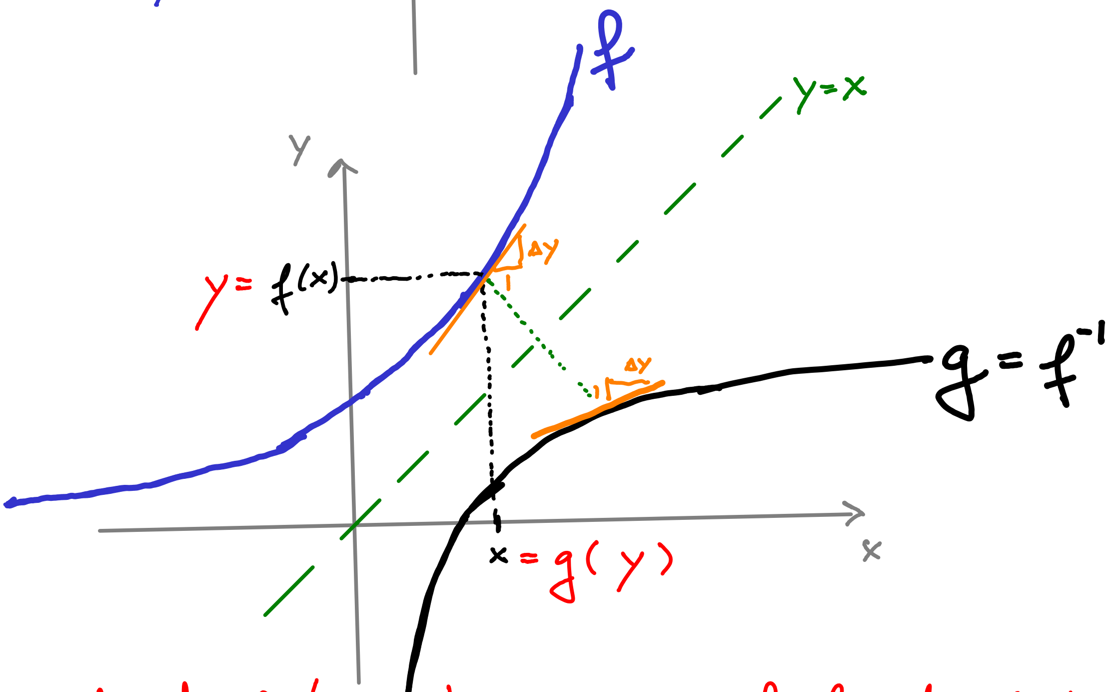
Inverse Functions

Definition: A function $f : [a,b] \to \mathbb{R}$, where $a, b \in \mathbb{R}$, $a < b$, is *invertible* if there exists a function g s.t. $g(f(x)) = x$ for all $x \in [a,b]$.

Note, for such a g to exist, f must satisfy the condition that $f(x_1) = f(x_2)$ implies $x_1 = x_2$, in which case f is called *injective*.

Another way of saying this is that every horizontal line meets the graph of f in at most one point.





The graph of f^{-1} is the graph of f reflected in $y=x$.

Derivative of inverse function

Fact: f has an inverse if f is injective.

Suppose f has an inverse, say f^{-1}, then

$f^{-1}(f(x)) = x$ and using the chain rule,

we get

$$(f^{-1})'(f(x))\, f'(x) = 1$$

Similarly, $f(f^{-1}(x)) = x$ and

$$f'(f^{-1}(x))\,(f^{-1})'(x) = 1$$

which says

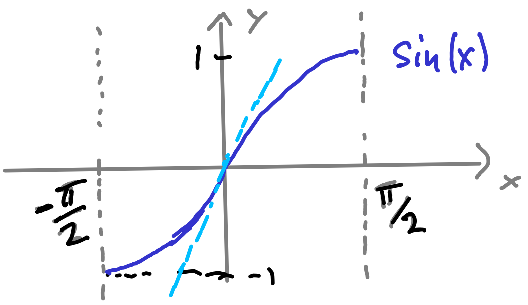
$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

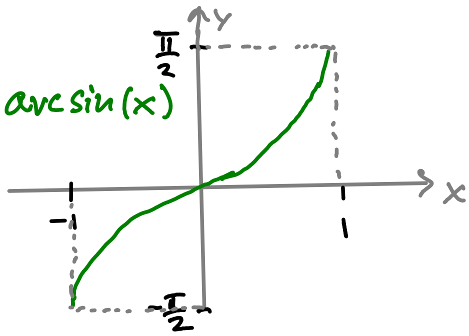
Example: ① Let $f(x) = e^x$, then $f^{-1}(x) = \ln(x)$

So $\frac{d}{dx}(\ln(x)) = \frac{1}{f'(\ln(x))} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$

which we already knew.

② let $f(x) = \sin(x)$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$





<u>Problem:</u> Find $\frac{d}{dx}\left(\arcsin(x)\right)$.

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\cos\left(\arcsin(x)\right)}$$

Recall <u>arcsin</u> is the inverse function of <u>sin</u>.

$$\begin{tikzcd} \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \arrow[r, bend left, "\sin"] & \left[-1, 1\right] \arrow[l, bend left, "\arcsin"] \end{tikzcd}$$

So $\arcsin(\sin(x)) = x$ and $\sin(\arcsin(x))$.

We found $\dfrac{d}{dx}\left(\arcsin(x)\right) = \dfrac{1}{\cos(\arcsin(x))}$.

The books say $\dfrac{d}{dx}\left(\arcsin(x)\right) = \dfrac{1}{\sqrt{1-x^2}}$.

What's going on here?

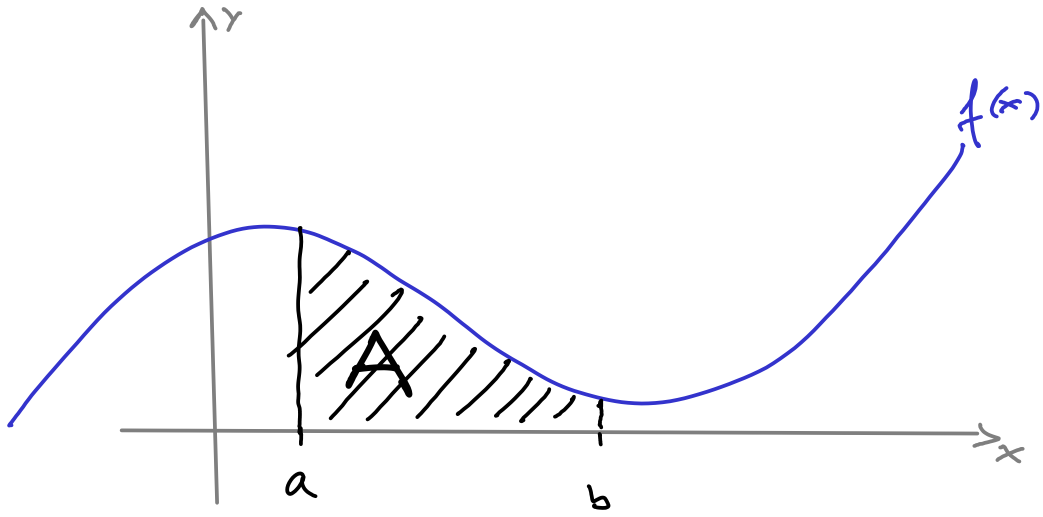
Note that for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$, we have $\cos(t) \geq 0$

So $\cos(t) = \sqrt{\cos^2(t)} = \sqrt{1-\sin^2(t)}$ and with $t = \arcsin(x)$

$$\cos(\arcsin(x)) = \sqrt{1-\sin^2(\arcsin(x))} = \sqrt{1-x^2}.$$

INTEGRATION

Integration can be seen as the inverse operation to differentiation, or as a limiting process for calculating areas, volumes, lengths of curves, work (pysics).



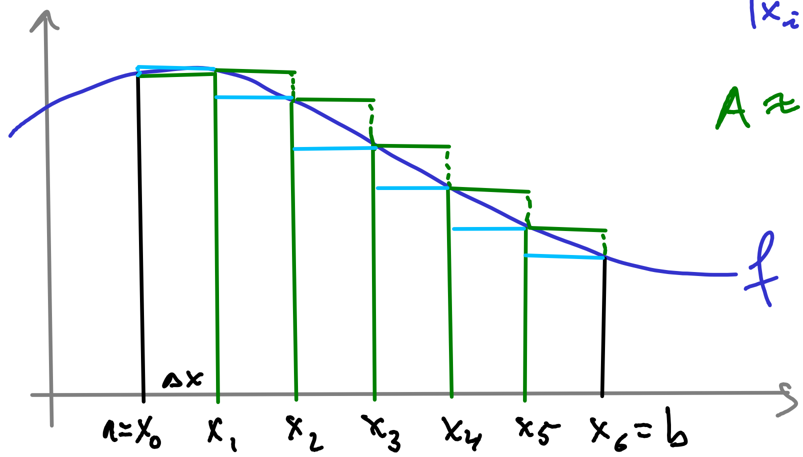
What is the area A below the graph of f ($f(x) \geq 0$), the x-axis and the two lines $x=a$ and $x=b$?

__Idea__: Split the area A into rectangles (approximately) of the same width. Then take the limit when the number of rectangles goes to infinity.

$|x_i - x_{i-1}| = \Delta x$

$A \approx f(x_0)\Delta x + f(x_1)\Delta x + \ldots$
$\ldots + f(x_5)\Delta x$

$A \approx f(x_1)\Delta x + \ldots + f(x_6)\Delta x$



On the interval $[a,b]$, if we choose N rectangles, then $\Delta x = \frac{b-a}{N}$ and the area is approximately

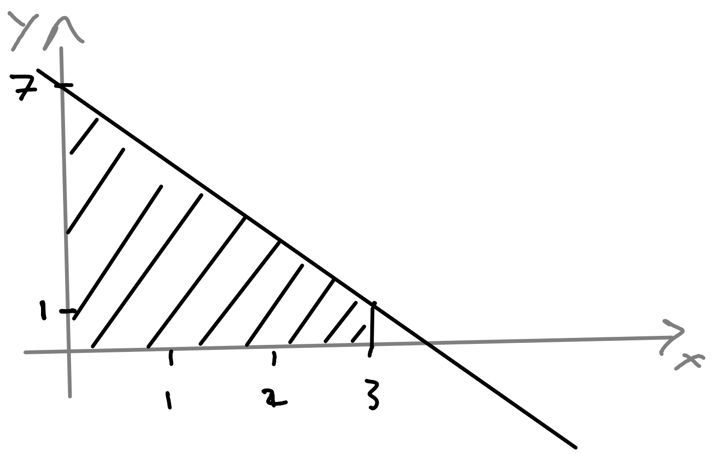
$$A \approx \Delta x\Big(f(a) + f(a+\Delta x) + f(a+2\Delta x) + \dots + f(a+(N-1)\Delta x)\Big)$$

$$= \Delta x \sum_{n=0}^{N-1} f(a+n\Delta x)$$

The limit when $N \to \infty$ is called <u>the integral of f from a to b</u> and written as

$$\int_a^b f(x)\,dx$$

<u>Example</u>: Find the area under the graph of $f(x) = -2x+7$ between $x=0$ and $x=3$



Here $b=3$, $a=0$. Let's start with $N=3$, then $\Delta x = 1$

$$A_3 \approx \Delta x\big(f(0)+f(1)+f(2)\big) = 7+5+3 = 15$$

$N = 6$, then $\Delta x = \frac{1}{2}$ and

$$A_6 \approx \frac{1}{2}\left(f(0) + f(\tfrac{1}{2}) + f(1) + f(\tfrac{3}{2}) + f(2) + f(\tfrac{5}{2})\right)$$

$$= \frac{1}{2}\left(f(0) + f(1) + f(2)\right) + \frac{1}{2}(6 + 4 + 2)$$

$$= \frac{15}{2} + 6 = 13.5$$

$-2x+7$

$N = 12$, then $\Delta x = \frac{1}{4}$

$$A_{12} = \frac{1}{2}A_6 + \frac{1}{4}\left(f(\tfrac{1}{4}) + f(\tfrac{3}{4}) + f(\tfrac{5}{4}) + f(\tfrac{7}{4}) + f(\tfrac{9}{4}) + f(\tfrac{11}{4})\right)$$

$$= 6.75 + \frac{1}{4}\left(\frac{13}{2} + \frac{11}{2} + \frac{9}{2} + \frac{7}{2} + \frac{5}{2} + \frac{3}{2}\right)$$

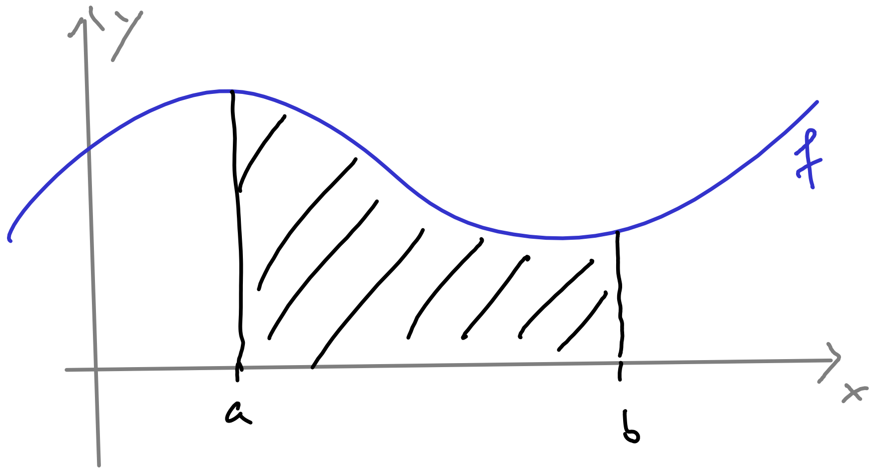
$$= 6.75 + \frac{1}{8}(48) = 12.75$$

Integration continued

Recall that we write

$$\int_a^b f(x)\,dx$$

for the definite integral from a to b of $f(x)$, which is defined as the limit of approximations of the area (for $f(x)>0$) between $x=a$, $x=b$, the x-axis and the graph of f.



The Fundamental Theorem of Calculus

For a continuous function f, we have

$$\frac{d}{dt}\left(\int_a^t f(x)\,dx\right) = f(t).$$

This means that integrating and then differentiating is the identity operation.

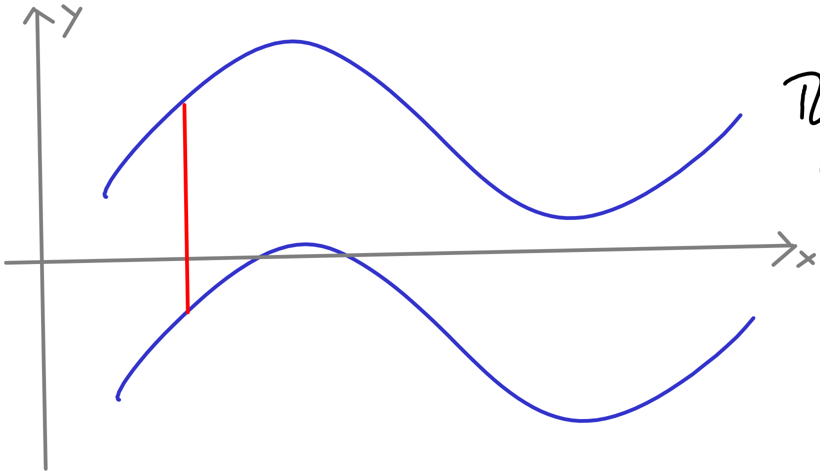
This already allows us to calculate some integrals.

<u>Definition</u>: Given a function $f(x)$, we say that $F(x)$ is an anti-derivative of $f(x)$ if $\frac{dF}{dx}(x) = f(x)$.

The Fundamental Theorem of Calculus says:

$$\int_a^x f(t)\,dt \quad \text{is an anti-derivative of } f.$$

Note: Since derivative is slope, there are many anti-derivatives for a given f.



These two graph have the same slope at every x.

Any two anti-derivatives of f differ by a constant.

We write $\int f(x)\,dx$ for an anti-derivative of f.

Basic integrals

$f(x)$	$\int f(x)$	
x^n	$\frac{1}{n+1} x^{n+1}$	for $n \neq -1$
x^{-1}	$\ln\lvert x\rvert$	(by definition)
$\sin(x)$	$-\cos(x)$	
$\cos(x)$	$\sin(x)$	
e^x	e^x	
$\ln(x)$	$x\ln(x) - x$	(came from MAXIMA)

Check: $\frac{d}{dx}(x\ln(x) - x)$

$= \ln(x) + x\frac{1}{x} - 1 = \ln(x)$.

Basic Rules

Let f and g be functions, and C a constant

(1) $\int (f(x) + g(x))\,dx = \int f(x)dx + \int g(x)dx$

(2) $\int C f(x)\,dx = C\int f(x)\,dx$

(3) For $a < b < c$: $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

④ If $F(x) = \int f(x)\,dx$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

<u>Examples</u>: Find $\int_0^3 -2x+7\,dx$

$$\int -2x+7\,dx = -x^2+7x+c = F(x)$$

a constant

$$\int_0^3 -2x+7\,dx = \Big[-x^2+7x+c\Big]_0^3$$

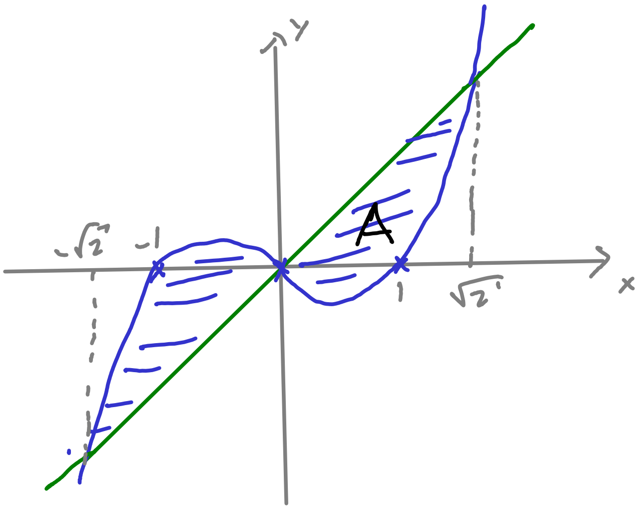
$$= F(3) - F(0)$$

$$= (-9+21+c) - (c)$$

$$= 12$$

Applications of integrals

Problem: Calculate the bounded area between the graph of $f(x) = x^3 - x$ and the line $y = x$.



$f = x^3 - x = x(x^2-1) = x(x+1)(x-1)$

$f' = 3x^2 - 1$

Points of intersection:

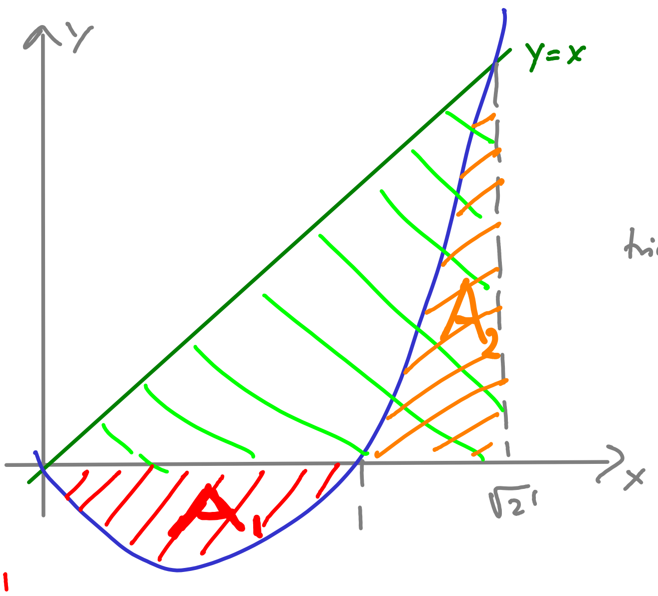
$f(x)$ and $g(x)$ intersect when $f(x) = g(x)$.

Here $x^3 - x = x$ or $x^3 - 2x = 0$

or $x(x^2 - 2) = 0$, so $x = 0$ or $x = \pm\sqrt{2}$

Hence the area is twice (by symmetry) the area between the graphs and $x = 0$ and $x = \sqrt{2}$.

Remember, area is only given by an integral if $f(x) \geq 0$



We want

$T - A_2 + A_1$

triangle

$$A_2 = \int_1^{\sqrt{2}} (x^3 - x)\,dx$$

$$= \left[\tfrac{1}{4}x^4 - \tfrac{1}{2}x^2\right]_1^{\sqrt{2}}$$

$$= 0 - \left(\tfrac{1}{4} - \tfrac{1}{2}\right) = \tfrac{1}{4}$$

$$A_1 = -\int_0^1 x^3 - x\,dx = -\left[\tfrac{1}{4}x^4 - \tfrac{1}{2}x^2\right]_0^1$$

$$= -\left(\tfrac{1}{4} - 0\right) = \tfrac{1}{4}$$

The triangle has are $\frac{1}{2}\sqrt{2}^{\,2} = 1$, which also is the are of the bonded piece, ad the are in question is twice this, i.e. 2.

Alternatively: Notice that for $0 \le x \le \sqrt{2}$, the graph of $y = x$ is always above the graph of $y = x^3 - x$. This means that $x \ge x^3 - x$ for these values of x. In other words, $2x - x^3 \ge 0$ and the area of on piece can be calculated as

$$\int_0^{\sqrt{2}} (2x - x^3)\, dx = \left[x^2 - \frac{1}{4}x^4 \right]_0^{\sqrt{2}} = 2 - 1 - 0 = 1.$$

In general: The are between two consecutive intersections of $f(x)$ and $g(x)$, at $x = a$ ad $x = b$ say, is given by

$$\int_a^b f(x) - g(x)\, dx \qquad \text{if} \quad f(x) \ge g(x)$$

or by

$$\int_a^b g(x) - f(x)\, dx \qquad \text{if} \quad g(x) \ge f(x).$$

Remark: If we interpret $\int_a^b f(x)\,dx$ as area, then area below the x-axis is counted negatively.

Techniques for Integration

Using the chain rule we find

$$\frac{d}{dx}\left(\ln\left(f(x)\right)\right) = \frac{1}{f(x)}\frac{df}{dx}(x) = \frac{f'(x)}{f(x)}.$$

So we find that

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left(f(x)\right) + c$$

Example: Find $\displaystyle\int \frac{6x^2-8x+2}{x^3-2x^2+x+3}\,dx$.

Sol^n: $$\int \frac{6x^2-8x+2}{x^3-2x^2+x+3}\,dx = 2\int \frac{3x^2-4x+1}{x^3-2x^2+x+3}\,dx$$

$$= 2\ln\left(x^3-2x^2+x+3\right) + c$$

Problem: Find $\displaystyle\int \frac{3}{x^2+2x-3}\,dx$.

Sol^n: $$\frac{3}{x^2+2x-3} = \frac{3}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$$

$$= \frac{A(x+3)}{(x-1)(x+3)} + \frac{B(x-1)}{(x-1)(x+3)}$$

$$= \frac{(A+B)x + 3A - B}{(x-1)(x+3)}$$

Since we want both sides to be equal, we need $(A+B)x+3A-B=3$, which means

$$A+B=0 \quad \text{and} \quad 3A-B=3$$

$$B=-A \longrightarrow 4A=3$$

$$B=-3/4 \longleftarrow A=3/4$$

Now we win:

$$\int \frac{3}{x^2+2x-3}\,dx = \int \frac{3/4}{x-1} - \frac{3/4}{x+3}\,dx$$

$$= 3/4 \int \frac{1}{x-1}\,dx - 3/4 \int \frac{1}{x+3}\,dx$$

$$= 3/4 \ln|x-1| - 3/4 \ln|x+3| + c$$

$$= 3/4 \left(\ln|x-1| - \ln|x+3| \right) + c$$

$$= 3/4 \ln\left(\frac{|x-1|}{|x+3|}\right) + c$$

This trick is known as the partial fractions method.

Integration by parts

Using the product rule (for differentiation), we find

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}.$$

Now integrate both sides:

$$fg = \int \left(g\frac{df}{dx} + f\frac{dg}{dx}\right)dx$$

$$= \int g\frac{df}{dx}\,dx + \int f\frac{dg}{dx}\,dx$$

$$= \int g f'\,dx + \int f g'\,dx$$

From this we get that

$$\int f g'\,dx = fg - \int f' g\,dx.$$

Example: Find $\int x \ln(x)\,dx$.

Sol^n: Let $f(x) = \ln(x)$ and $g'(x) = x$. Then

$$f'(x) = \frac{1}{x} \quad \text{and} \quad g(x) = \frac{1}{2}x^2 \quad \text{and}$$

$$\int x \ln(x) = \frac{1}{2}x^2 \ln(x) - \int \frac{1}{x}\frac{1}{2}x^2\,dx$$

$$= \frac{1}{2}x^2 \ln(x) - \frac{1}{2}\int x\,dx$$

$$= \frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + c$$

Check: $\frac{d}{dx}\left(\frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + c\right)$

$$= x\ln(x) + \cancel{\frac{1}{2}x^2\frac{1}{x}} - \cancel{\frac{1}{2}x} = x\ln(x).$$

Problem: Find $\int x e^x\,dx$.

Sol^n: Put $f(x) = x$ and $g'(x) = e^x$

Then $f'(x) = 1$ and $g(x) = e^x$.

So $\int x e^x\,dx = xe^x - \int e^x\,dx = e^x(x-1)$

More on partial fractions

Fundamental Theorem of Algebra

Let $p(x)$ be a polynomial of degree at least one. Then

(i) There is a root/zero of $p(x)$ in $\mathbb{C}$, the complex numbers.

(ii) If a is a root of $p(x)$, i.e. $p(a)=0$, then $p(x) = (x-a)\,q(x)$, where $q(x)$ is a polynomial of degree one less than p.

Consequence: Over $\mathbb{C}$, every polynomial is a product of linear factors, that is of factors of the form $(x-a)$, where $a = \text{const}$.

Note: This is not true over $\mathbb{R}$, because some quadratic polynomials without roots.

Example: $p(x) = x^2+1$ has no real root.

In the context of partial fractions, this means we need to deal with the case where the denominator has such an irreducible quadratic factor.

Example: Write $\dfrac{1}{(x-1)(x^2+2)}$ as partial fraction.

Sol^n: Want
$$\frac{1}{(x-1)(x^2+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2}$$

Now
$$\frac{A}{x-1} + \frac{Bx+C}{x^2+2} = \frac{A(x^2+2)+(Bx+C)(x-1)}{(x-1)(x^2+2)}$$
$$= \frac{(A+B)x^2+(C-B)x+2A-C}{(x-1)(x^2+2)}$$

So $A+B=0$, $C-B=0$, $2A-C=1$, and

$B=-A$, $B=C$, and then $3A=1$.

Hence:
$$\frac{1}{(x-1)(x^2+1)} = \frac{1}{3}\,\frac{1}{x-1} - \frac{1}{3}\,\frac{x+1}{x^2+2}$$
$$= \frac{1}{3}\left(\frac{1}{x-1} - \frac{x+1}{x^2+2}\right)$$
$$= \frac{1}{3}\;\frac{x^2+2-(x+1)(x-1)}{(x-1)(x^2+2)}$$

Now we can calculate
$$\int \frac{1}{(x-1)(x^2+1)}\,dx = \frac{1}{3}\int\left(\frac{1}{x-1} - \frac{x+1}{x^2+2}\right)dx$$
$$= \frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{x}{x^2+2}\,dx - \frac{1}{3}\int \frac{1}{x^2+2}\,dx$$
$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\int \frac{2x}{x^2+2}\,dx - \frac{1}{3}\int \frac{1}{x^2+2}\,dx$$
$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+2) - \frac{1}{3}\int \frac{1}{x^2+2}\,dx$$

From the tables we find

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) \qquad (a>0).$$

So

$$\int \frac{1}{x^2+2}\,dx = \int \frac{1}{x^2+\sqrt{2}^{\,2}}\,dx = \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)$$

and

$$\int \frac{1}{(x-1)(x^2+2)}\,dx = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+2) - \frac{1}{3\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + c$$

Long division with polynomials

Factorising polynomials of (high?) degree can be done by iterating

1. Find a zero, say a, of $p(x)$.
2. Find $q(x)$ s.t. $p(x) = q(x)(x-a)$

Example: Factorise $p(x) = x^3-2x^2-x+2$.

Trial and error shows that $x=1$ is a root.

$$(x^3-2x^2-x+2)/(x-1) = x^2-x-2$$

$$\begin{array}{l} x^3-x^2 \\ \hline -x^2-x+2 \\ -x^2+x \\ \hline -2x+2 \\ -2x+2 \\ \hline 0 \end{array}$$

So $x^3-2x^2-x+2 = (x-1)(x^2-x-2)$

$$(x^2-x-2)/(x-2) = x+1$$

$$\begin{array}{l} x^2-2x \\ \hline x-2 \end{array}$$

Now $x^3-2x^2+x+2 = (x-1)(x-2)(x+1)$.

More integration

Problem: Find $\int \cos^2(x)\,dx$.

Sol^n: Use the addition theorem for cosine to get

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$

Then $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$

$$= 2\cos^2(\alpha) - (\cos^2(\alpha) + \sin^2(\alpha))$$

$$= 2\cos^2(\alpha) - 1$$

So $\cos^2(\alpha) = \frac{1}{2}(\cos(2\alpha)+1)$, and hence

$$\int \cos^2(x)\,dx = \frac{1}{2}\int (\cos(2x)+1)\,dx$$

$$= \frac{1}{4}\sin(2x) + \frac{1}{2}x + c$$

Problem: Find $\int \underbrace{\sin(x)}_{f}\underbrace{\cos(x)}_{g'}\,dx$

$\int fg'\,dx = fg - \int f'g\,dx$

Use integration by parts:

$$\int \sin(x)\cos(x)\,dx = \sin^2(x) - \int \cos(x)\sin(x)\,dx$$

$$2\int \sin(x)\cos(x)\,dx = \sin^2(x)$$

$$\int \sin(x)\cos(x)\,dx = \frac{1}{2}\sin^2(x) + c$$

Substitution

Recall the chain rule:

$$\frac{d}{dx}\Big(f(g(x)) \Big) = f'(g(x))\, g'(x)$$

Integrating both sides gives

$$f(g(x)) = \int f'(g(x))\, g'(x)\, dx$$

$$= \int f'(g)\, dg \qquad\qquad \frac{dg}{dx} = g'(x)$$

How is it used?

Let's do $\int \cos(x)\sin(x)\,dx$ again.

Put $g = \sin(x)$. Then $dg = \cos(x)\,dx$ and

$$\int \sin(x)\cos(x)\,dx = \int g\, dg = \frac{1}{2} g^2 + c$$

$$= \frac{1}{2}\sin^2(x) + c.$$

Another example: Find $\int x e^{x^2+2}\,dx$

Put $g = x^2+2$. Then $dg = 2x\,dx$ and

$$\int x e^{x^2+2}\,dx = \int \frac{1}{2} e^{g}\,dg = \frac{1}{2} e^{g} + c = \frac{1}{2} e^{x^2+2} + c.$$

Problem: Find $\int \frac{1}{a^2+x^2}\,dx$, where $a>0$.

Put $x = a\tan(u)$. Then $\frac{dx}{du} = \frac{d}{du}\left(\frac{\sin(u)}{\cos(u)}\right)$

$u = \arctan\left(\frac{x}{a}\right)$

$$= \frac{\cos^2(u)+\sin^2(u)}{\cos^2(u)} = \frac{1}{\cos^2(u)}$$

So $dx = \frac{a}{\cos^2(u)}\,du$ and

$$\int \frac{1}{a^2+x^2}\,dx = \int \frac{1}{a^2+a^2\tan^2(u)}\,\frac{a}{\cos^2(u)}\,du$$

$$= \int \frac{1}{a^2\left(1+\frac{\sin^2(u)}{\cos^2(u)}\right)}\,\frac{a}{\cos^2(u)}\,du$$

$$= \frac{1}{a}\int \frac{1}{\frac{\cos^2(u)+\sin^2(u)}{\cos^2(u)}}\cdot\frac{1}{\cos^2(u)}\,du$$

$$= \frac{1}{a}\int du = \frac{1}{a}u + c$$

$$= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c.$$

One more: Find $\int \sqrt{1-x^2}\,dx$

Put $x = \sin(u)$. Then $u = \arcsin(x)$ and

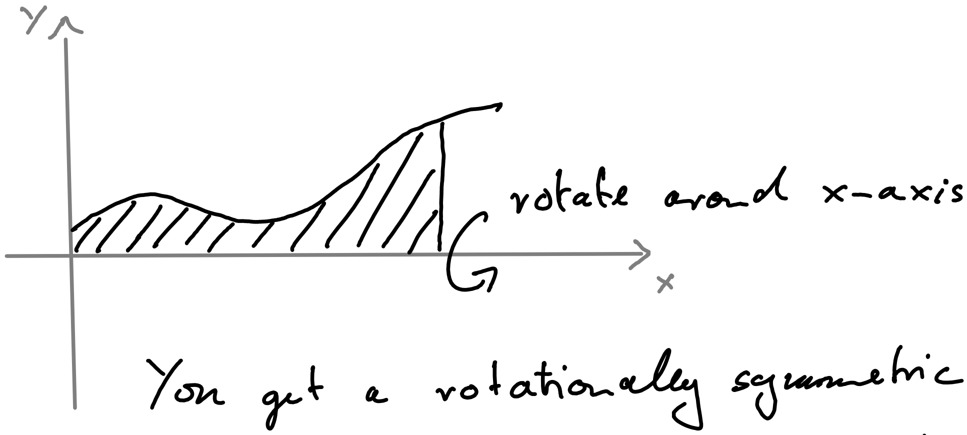
$dx = \cos(u)\,du$ and

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2(u)}\,\cos(u)\,du = \int \cos^2(u)\,du$$

$$= \frac{1}{4}\sin(2u) + \frac{1}{2}u + c = \frac{1}{4}\sin\left(2\arcsin(x)\right) + \frac{1}{2}\arcsin(x) + c.$$

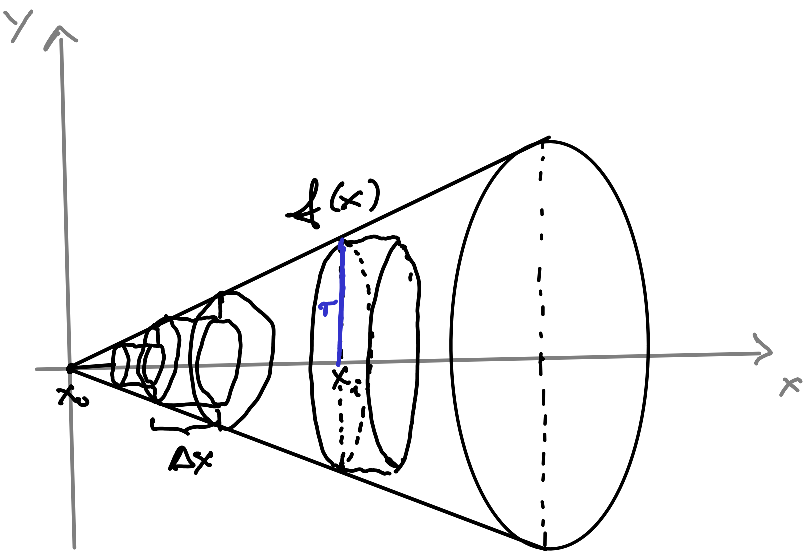
Applications of Integration

Volumes of revolution



You get a rotationally symmetric object in 3D and one can ask for its volume.

The idea is to slice the object into approximately discs of thickness Δx and then the volume is the sum of the volumes of those disks and we take the limit $\Delta x \to 0$.



Each disc has volume $\pi [f(x_i)]^2 \Delta x$ where $x_i = x_0 + i \Delta x$.

So the volume is approximated by

$$V \approx \sum_{i=0}^{n-1} \Delta x \, \pi \left[f(x_i) \right]^2$$ if we have n discs.

In the limit we get

$$V = \int_a^b \pi [f(x)]^2 \, dx$$, where a and b are the left and right boundaries.

Example: Calculate the volume of the cone of height 6 generated by revolving $f(x) = \frac{1}{2}x$ about the x-axis.

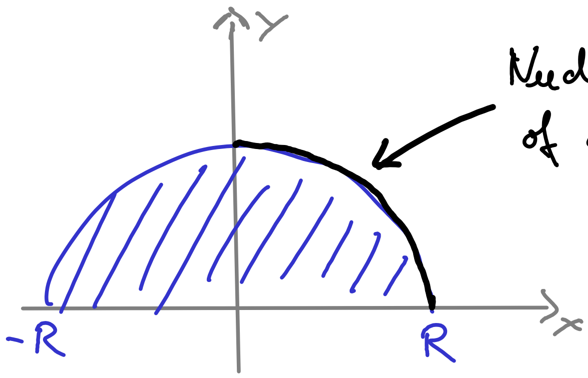
Solⁿ

$$\pi \int_0^6 \left(\frac{1}{2}x\right)^2 dx = \pi \int_0^6 \frac{1}{4}x^2 \, dx = \pi \left[\frac{1}{12}x^3\right]_0^6$$

$$= 18\pi$$

Example: Calculate the volume of a sphere of radius R.

Solⁿ: Rotate the half disc of radius R about the x-axis.



Need this as the graph of a function $f(x)$

Maybe be lazy and only calculate half the volume by only considering the quarter disc with $x \geq 0$.

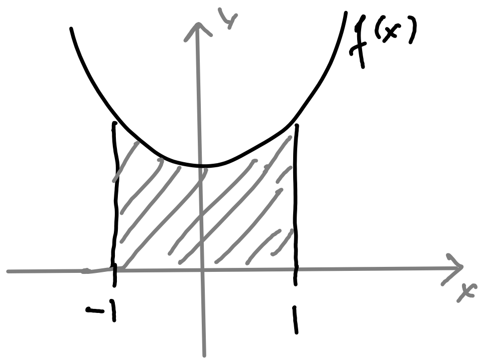
$x^2 + y^2 = R^2$ has as solutions the points (x, y) on the circle.

So $y = f(x) = \sqrt{R^2 - x^2}$ and the volume of the sphere is

$$V = 2\pi \int_0^R (R^2 - x^2)\, dx = 2\pi \left[R^2 x - \frac{1}{3} x^3 \right]_0^R$$

$$= 2\pi \left(R^3 - \frac{1}{3} R^3 \right) = \frac{4}{3} \pi R^3$$

Last example: Find the volume of the solid of revolution obtained by rotating the are under $f(x) = x^2 + 2$ between $x = -1$ and $x = 1$ about the x-axis

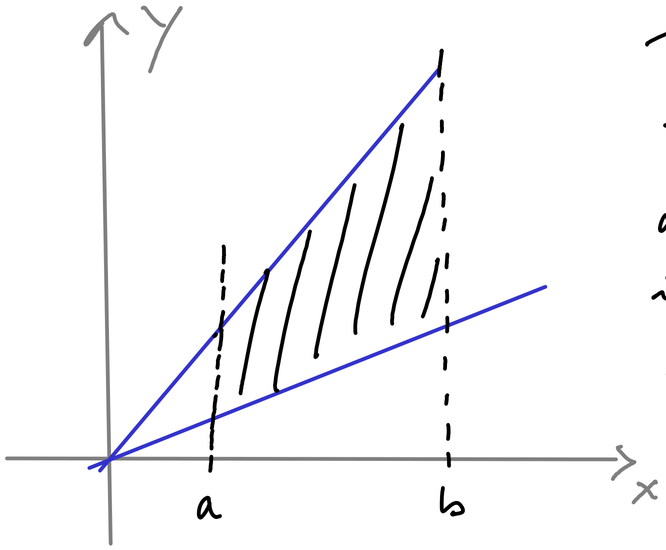


$$V = \pi \int_{-1}^{1} (x^2+2)^2\, dx = \pi \int_{-1}^{1} (x^4 + 4x^2 + 4)\, dx$$

$$= \pi \left[\frac{1}{5} x^5 + \frac{4}{3} x^3 + 4x \right]_{-1}^{1} = \pi \left(\frac{1}{5} + \frac{4}{3} + 4 - \left(-\frac{1}{5} - \frac{4}{3} - 4 \right) \right)$$

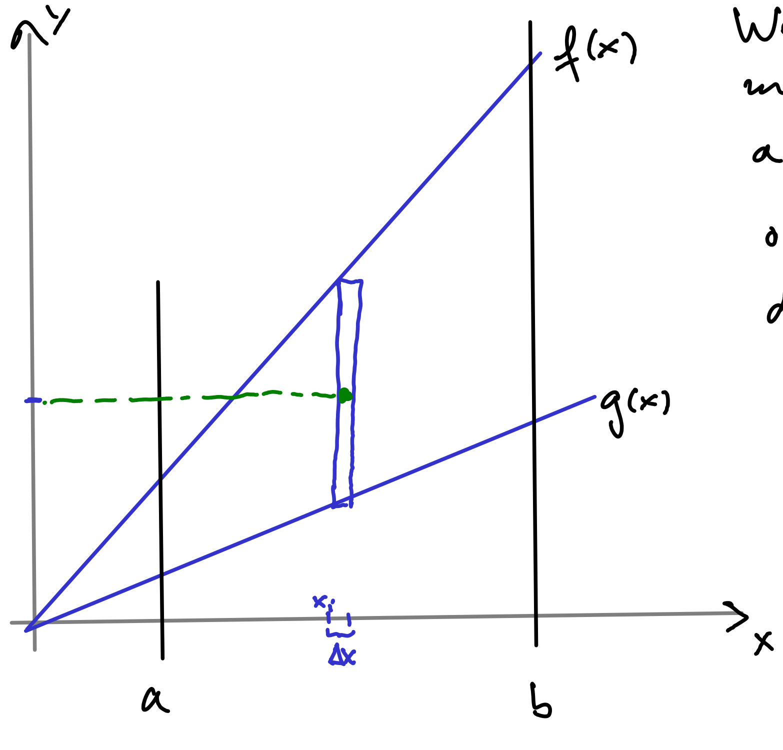
$$= 2\pi \left[\frac{3 + 20 + 60}{15} \right] = \frac{166}{15} \pi = 11\pi + \frac{1}{15} \pi$$

Centroids aka centre of mass



The centre of mass of an 2D object is that point at which you can ballance it on a pencil.
(True for convex objects.)

More precisely it is the point about which no momentum is generated by the object.



We'll calculate the momentum about each axis, assuming the object has homogeneous density ρ.

Moment about the x-axis: 1 For a thin strip:

$$\underbrace{\frac{g(x_i)+f(x_i)}{2}}_{\text{distance of mid point from x-axis}} \underbrace{\left(f(x_i)-g(x_i)\right)\Delta x\,\rho}_{\text{mass}}$$

② Summing these for n strips and taking the limit when $n \to \infty$ we get

$$\frac{\rho}{2}\int_a^b (f(x))^2 - (g(x))^2 \, dx$$

Moment about the y-axis: ① For the same strip

$$\underset{\text{distance from y-axis}}{x_i} \; \underbrace{(f(x_i) - g(x_i))\Delta x \, \rho}_{\text{mass}}$$

② Again taking the limit, we get

$$\rho \int_a^b x\,(f(x) - g(x))\,dx$$

The whole object has mass

$$M = \rho \int_a^b (f(x) - g(x))\,dx \qquad (\text{area} \times \text{density} = \text{mass})$$

The coordinates of the centre of mass, say $\bar{x}$ and $\bar{y}$, must satisfy: $\bar{x}M$ = moment about y-axis

$\bar{y}M$ = moment about x-axis

Hence
$$\bar{x} = \frac{\rho}{M}\int_a^b (f(x) - g(x))\,x\,dx$$

$$\bar{y} = \frac{\rho}{2M}\int_a^b \left((f(x))^2 - (g(x))^2\right)dx$$

We can cancel ϱ here to get

$$\bar{x} = \frac{1}{A} \int_a^b x \left(f(x) - g(x) \right) dx$$

and

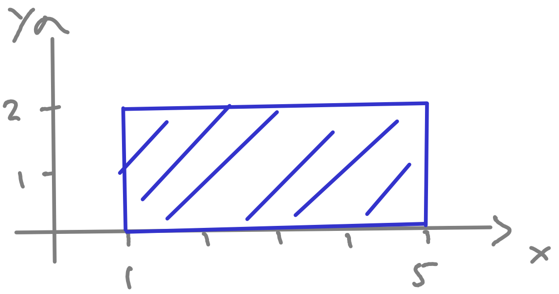
$$\bar{y} = \frac{1}{2A} \int_a^b \left(f(x)^2 - g(x)^2 \right) dx ,$$

where A is the area of the object:

$$A = \int_a^b \left(f(x) - g(x) \right) dx .$$

Examples of centroids

Problem: Find the centre of mass of the rectangle $[1,5]\times[0,2]$



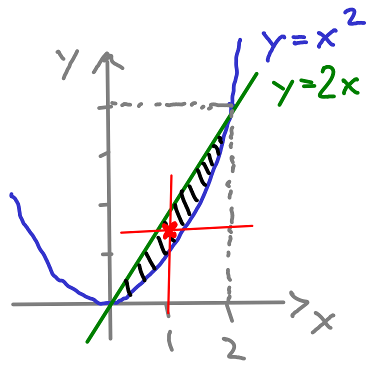
Sol^n: The area is $A=8$.

$$\bar{x} = \frac{1}{A}\int_1^5 2x\,dx = \frac{1}{8}\left[x^2\right]_1^5 = \frac{1}{8}(25-1) = 3$$

$$\bar{y} = \frac{1}{2A}\int_1^5 4\,dx = \frac{1}{16}\left[4x\right]_1^5 = \frac{1}{16}(20-4) = 1.$$

Problem: Find the centre of mass of the bounded area between the curves $y=x^2$ and $y=2x$.

Sol^n:



① The area is

$$A = \int_0^2 (2x - x^2)\,dx = \left[x^2 - \frac{1}{3}x^3\right]_0^2$$
$$= 4 - \frac{8}{3} = \frac{4}{3}$$

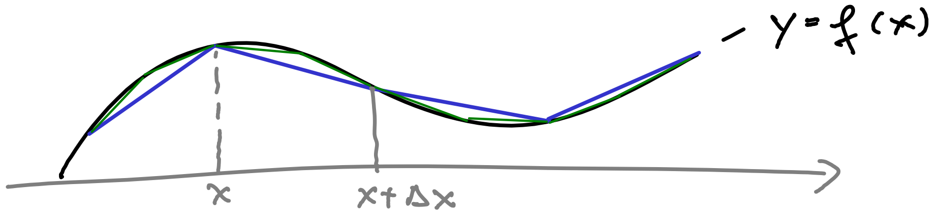
② $$\bar{x} = \frac{1}{A}\int_0^2 x(2x-x^2)\,dx = \frac{3}{4}\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2$$
$$= \frac{3}{4}\left(\frac{16}{3} - 4\right) = 1$$

$$\bar{y} = \frac{1}{2A}\int_0^2 \left((2x)^2 - x^4\right)dx = \frac{3}{8}\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2$$
$$= \frac{3}{8}\left(32\left(\frac{1}{3}-\frac{1}{5}\right)\right) = \frac{24}{15} = \frac{8}{5}$$

Lengths of Curves

How long is the graph of the sine wave from 0 to π?

Or a piece of a parabola?



Idea: Approximate the curve by short line segments,

say from $(x, f(x))$ to $(x+\Delta x, f(x+\Delta x))$, sum them up and then take the limit when $\Delta x \to 0$, i.e. the number of pieces goes to infinity

The length of the line segment from $(x, f(x))$ to $(x+\Delta x, f(x+\Delta x))$ is

$$\sqrt{(\Delta x)^2 + (f(x+\Delta x) - f(x))^2}$$

$$= \sqrt{(\Delta x)^2 \left(1 + \left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right)^2\right)}$$

$$= \Delta x \sqrt{1 + \left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right)^2} \xrightarrow[\Delta x \to 0]{} \Delta x \sqrt{1 + [f'(x)]^2}$$

Summing all the lengths of the pieces and taking the limit results in the formula for the length:

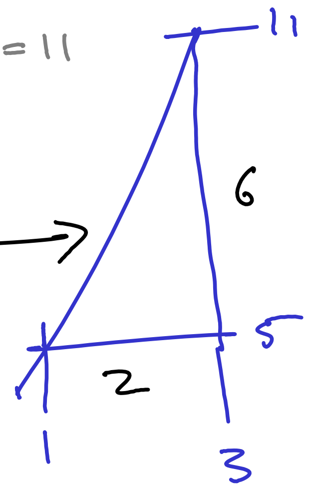
$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Easy example: Find the length of $f(x) = 3x + 2$ between $x = 1$ and $x = 3$.

Sol^n: $L = \int_1^3 \sqrt{1 + 3^2}\, dx = \sqrt{10}\,[x]_1^3 = 2\sqrt{10}$

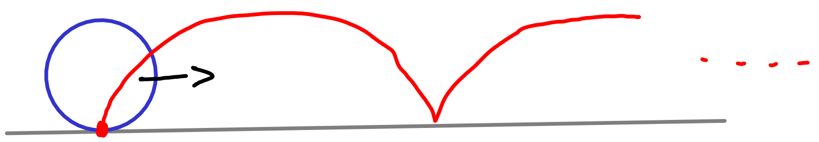
Check: $f(1) = 5$, $f(3) = 11$

$\sqrt{40} = 2\sqrt{10}$



Warning: Often the integral one gets is near impossible to work out!

One example you might try is the cycloid, the curve described by a point on a wheel rolling along



Calculating Arc Length

Example: Find the length of the curve $y = x^2 - \frac{1}{8}\ln(x)$ for $1 \le x \le 2$.

Sol^n: The length is

$$\int_1^2 \sqrt{1 + \left(2x - \frac{1}{8x}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}}\, dx$$

$$= \int_1^2 \sqrt{4x^2 + \frac{1}{2} + \frac{1}{64x^2}}\, dx = \int_1^2 \sqrt{\left(2x + \frac{1}{8x}\right)^2}\, dx$$

$$= \int_1^2 \left(2x + \frac{1}{8x}\right) dx = \left[x^2 + \frac{1}{8}\ln(x)\right]_1^2 = 4 + \frac{1}{8}\ln(2) - (1+0)$$

$$= 3 + \frac{1}{8}\ln(2)$$

Example: Find the length of $y = \frac{2}{3}(x-1)^{3/2}$ for $1 \le x \le 4$.

Sol^n: $y' = (x-1)^{1/2}$, so the length is

$$\int_1^4 \sqrt{1 + (x-1)}\, dx = \int_1^4 \sqrt{x}\, dx = \left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$$

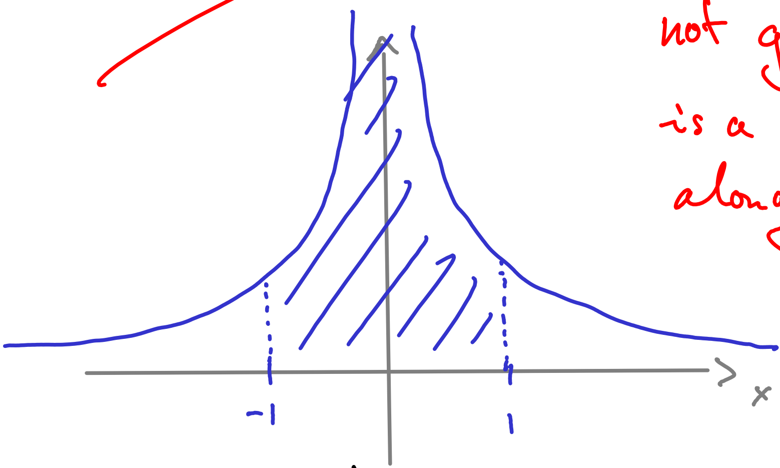
Improper Integrals

An integral in which one or both boundaries are infinite or discontinuities of the function to be integrated are called <u>improper</u>. They may or may not exist.

<u>Problem</u>: Find the are under the graph of $f(x) = \frac{1}{x^2}$ between $-1 \leq x \leq 1$.

~~Fine, we calculate~~ $\cancel{\int_{-1}^{1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-1}^{1} = -1-1=-2.}$

not good! There is a discontinuity along the way.



<u>Solution</u>: Take $\int_{\varepsilon}^{1} \frac{1}{x^2}\,dx$ then let $\varepsilon \to 0$. $\varepsilon > 0$

$$\int_{\varepsilon}^{1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{\varepsilon}^{1} = -1 + \frac{1}{\varepsilon} \xrightarrow{\varepsilon \to 0} \infty$$

<u>Conclusion</u>: Even half the area is not finite!

Another example: What about the area under $y = \frac{1}{x^2}$ between 1 and ∞?

Idea: Calculate $\int_1^L \frac{1}{x^2}\,dx$ and then let $L \to \infty$.

$$\int_1^L \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^L = -\frac{1}{L} + 1 \xrightarrow{L\to\infty} 1$$

Convention: We write $\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^b f(x)\,dx$

or even $\int_{-\infty}^\infty f(x)\,dx$ in order to shorten

$$\lim_{L\to\infty} \int_a^L f(x)\,dx, \quad \lim_{L\to-\infty} \int_{-\infty}^b f(x)\,dx \quad \text{or} \quad \lim_{L\to\infty} \int_{-L}^L f(x)\,dx.$$

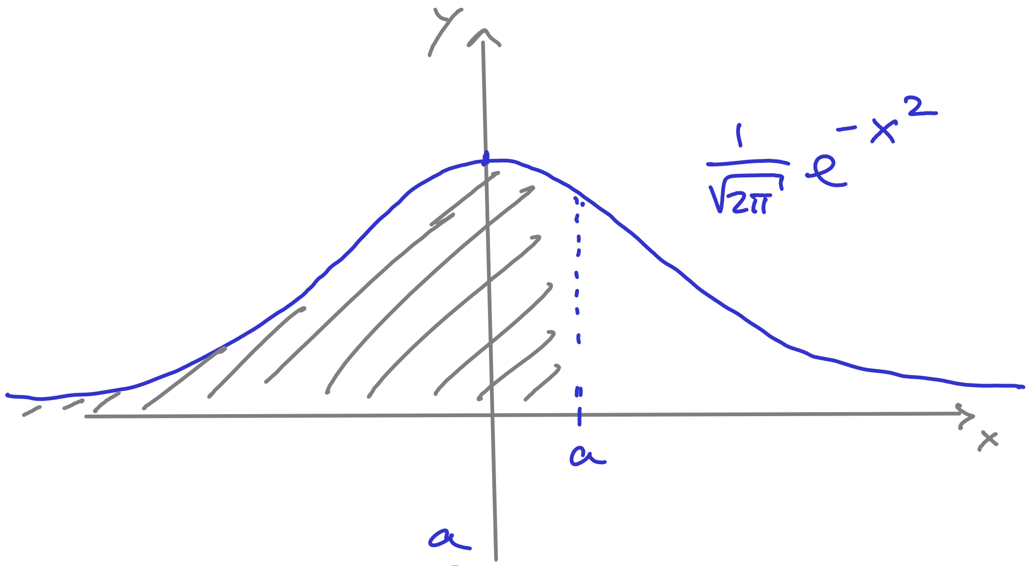
Note: Being a limit, these integrals may not exist.

Example: $\int_1^\infty \frac{1}{x}\,dx = \left[\ln(x)\right]_1^\infty = \lim_{L\to\infty} \ln(L) - 0$

does not exist.

<u>Example:</u> $\int_{-\infty}^{0} e^x \, dx = \left[e^x \right]_{-\infty}^{0} = 1 - \lim_{L \to -\infty} e^L = 1$

In probability a density function $f : \mathbb{R} \to \mathbb{R}$ is a function with $\int_{-\infty}^{\infty} f(x)\,dx = 1$.



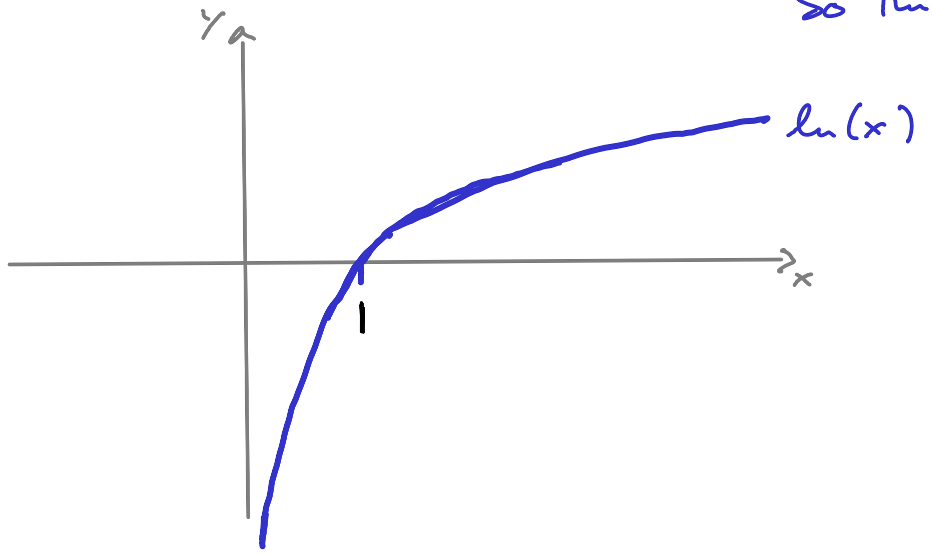
$$P(X \leq a) = \int_{-\infty}^{a} f(x)\,dx$$

Some more improper integrals

Problem: Does $\displaystyle\int_0^1 \frac{1}{x}\,dx$ exist?

Sol^n: $\displaystyle\int_\varepsilon^1 \frac{1}{x}\,dx = \Big[\ln(x)\Big]_\varepsilon^1 = 0 - \ln(\varepsilon) \xrightarrow{\varepsilon \to 0} \infty$

So this integral does not exist.



$$\lim_{x\to\infty} \ln(x) = \infty$$

Problem: Does $\displaystyle\int_2^\infty \frac{x^2-1}{x^3-3x+4}\,dx$ exist?

Substitution: $u = x^3 - 3x + 4$. So $\dfrac{du}{dx} = 3x^2 - 3 = 3(x^2-1)$

or $\dfrac{du}{3} = (x^2-1)\,dx$ and we get

$$\int \frac{x^2-1}{x^3-3x+4}\,dx = \int \frac{1}{3}\,\frac{1}{u}\,du = \frac{1}{3}\ln(u)$$

$$= \frac{1}{3}\ln(x^3 - 3x + 4)$$

So $\displaystyle\int_2^\infty \frac{x^2-1}{x^3-3x+4}\,dx = \Big[\frac{1}{3}\ln(x^3-3x+4)\Big]_2^\infty = \frac{1}{3}\lim_{L\to\infty} \ln(L^3-3L+4) - \frac{1}{3}\ln(6)$

Again, this integral does not exist.

<u>Problem</u>: What about $\displaystyle\int_2^\infty \frac{x}{x^3-3x-2}\,dx$?

Roughly speaking, for large x we integrate $\frac{1}{x^2}$, so we expect the integral to exist.

Partial fractions:

① Factorise x^3-3x-2.

-1 is a zero and we find

$$(x^3-3x-2)/(x+1) = x^2-x-2$$

$$\begin{array}{l} x^3+x^2 \\ \hline -x^2-3x-2 \\ -x^2-x \\ \hline -2x-2 \end{array}$$

$$x = \frac{1}{2} \pm \sqrt{\frac{1}{4}+2} = \begin{cases} 2 \\ -1 \end{cases}$$

So $x^3-3x-2 = (x+1)^2(x-2)$.

Now solve $\dfrac{Ax+B}{(x+1)^2} + \dfrac{C}{x-2} = \dfrac{x}{x^3-3x-2}$.

$$(Ax+B)(x-2) + C(x+1)^2 = x$$

Put $x=2$ to get $9C=2$, i.e. $C=\frac{2}{9}$

Put $x=-1$ to get $-3(B-A) = -1$, i.e. $B-A = \frac{1}{3}$

Put $x=0$ to get $-2B+C = 0$, so $2B = \frac{2}{9}$

and $B = \frac{1}{9}$, and hence $A = \frac{1}{9} - \frac{1}{3} = -\frac{2}{9}$

Check: $\frac{1}{9}\left(\frac{-2x+1}{(x+1)^2} + \frac{2}{x-2}\right)$

$$= \frac{1}{9}\;\frac{\cancel{-2x^2} + 5x \cancel{-2} + 2(\cancel{x^2} + 2x \cancel{+1})}{(x+1)^2\,(x-2)}$$

$$= \frac{1}{9}\;\frac{9x}{(x+1)^2(x-2)} = \frac{x}{(x+1)^2(x-2)}$$

So $\displaystyle\int_2^\infty \frac{x}{x^3-3x-2}\,dx = \frac{1}{9}\int_2^\infty\left(\frac{-2x+1}{(x+1)^2} + \frac{2}{x-2}\right)dx$

$$= \frac{1}{9}\int_2^\infty \frac{-2x+1}{(x+1)^2}\,dx + \frac{2}{9}\int_2^\infty \frac{1}{x-2}\,dx$$

$$= \frac{1}{9}\int_2^\infty \frac{-2x-2+3}{(x+1)^2}\,dx + \frac{2}{9}\Big[\ln(x-2)\Big]_2^\infty$$

$$= \quad \cdots \quad + \frac{2}{9}\left(\underbrace{\lim_{L\to\infty}(L-2)}_{\infty} - \underbrace{\lim_{\varepsilon\to 2^+}(\varepsilon-2)}_{-\infty}\right)$$

And we cannot decide this way!

Continuing yesterdays problem

Recall that we tried to decide wether

$$\int_2^\infty \frac{x}{x^3-3x-2}\,dx \quad \text{exists or not.}$$

Using partial fractions we got

$$\int_2^\infty \frac{x}{x^3-3x-2}\,dx = \frac{1}{9}\int_2^\infty \frac{-2x-2+3}{(x+1)^2} + \frac{2}{9}\int_2^\infty \frac{1}{x-2}\,dx$$

$$= \frac{1}{9}\Big[-\ln\big((x+1)^2\big)\Big]_2^\infty + \frac{1}{9}\int_2^\infty \frac{3}{(x+1)^2}\,dx + \Big[\frac{2}{9}\ln(x-2)\Big]_2^\infty$$

$$= \frac{1}{9}\Big[-2\ln(x+1) - \frac{3}{x+1} + 2\ln(x-2)\Big]_2^\infty$$

$$= \frac{1}{9}\Big[2\ln\Big(\frac{x-2}{x+1}\Big) - \frac{3}{x+1}\Big]_2^\infty$$

$\ln(a)-\ln(b)$
$=\ln\left(\frac{a}{b}\right)$

$$= \frac{1}{9}\Big[\lim_{L\to\infty}\Big(2\ln\big(\underbrace{\tfrac{L-2}{L+1}}_{\to 1}\big) - \underbrace{\frac{3}{L+1}}_{\to 0}\Big) - \lim_{\varepsilon\to 2^+}\Big(2\ln\Big(\frac{\varepsilon-2}{\varepsilon+1}\Big) - \frac{3}{\varepsilon+1}\Big)\Big]$$

$$= -\frac{1}{9}\underbrace{\lim_{\varepsilon\to 2^+}\Big(2\ln\Big(\frac{\varepsilon-2}{\varepsilon+1}\Big)\Big)}_{\text{does not exist.}} + \frac{1}{9}$$

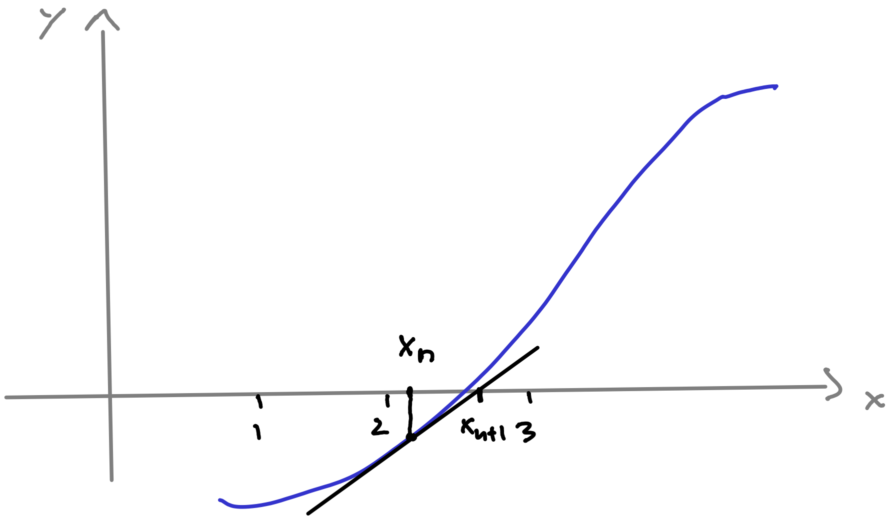
The problem is that roughly speaking we are integrating $\frac{1}{x^2}$ from 0 to ∞, which does not exist. Because 2 is the right most root of x^3-3x-2.

However, the calculation above shows that

$$\int_3^\infty \frac{x}{x^3-3x-2}\,dx = -\frac{1}{9}\ln\left(\frac{1}{4}\right) + \frac{1}{9}\cdot\frac{3}{4} = \frac{2}{9}\ln(2) + \frac{1}{12}$$

Numerical Methods

Newton's Method for finding zeros of $f(x)$.



This is iterative process calculating the next approximation x_{n+1} from the last approximation x_n.

1. Find the first approximation x_1 close to where the real zero is.
2. From x_n calculate x_{n+1} as the zero of the tangent line to the graph of f at the point $(x_n, f(x_n))$

The tangent line at x_n is given by

$$t(x) = f'(x_n)\,x + b \quad \text{s.t. } t(x_n) = f(x_n)$$

$$t(x_n) = f'(x_n)\,x_n + b = f(x_n)$$

So $b = f(x_n) - f'(x_n)\, x_n$ and

$$t(x) = f'(x_n)\, x + f(x_n) - f'(x_n)\, x_n$$

Hence $t(x) = 0$ implies

$$x = \frac{f'(x_n)\, x_n - f(x_n)}{f'(x_n)} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This gives Newton's approximation scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example: let $f(x) = x^3 - 2$. Find its root up to 4 decimal places.

Sol^n: $f'(x) = 3x^2$. Also $f(1) = -1$ and $f(\frac{3}{2}) = \frac{11}{8}$,
So the zero of f is between 1 and $3/2$.

Here $x_{n+1} = x_n - \dfrac{x_n^3 - 2}{3x_n^2} = \dfrac{2(x_n^3 + 1)}{3x_n^2}$

Choose $x_1 = 1$. Then $x_2 = \dfrac{4}{3}$, $x_3 = \dfrac{2\left(\frac{64}{27} + 1\right)}{\frac{16}{3}} = 1.263\overline{8}$

$x_4 = 1.2599$. Not much happens after this.

Numerical Integration

Recall the very first attempts at integration, using small rectangles to approximate area.

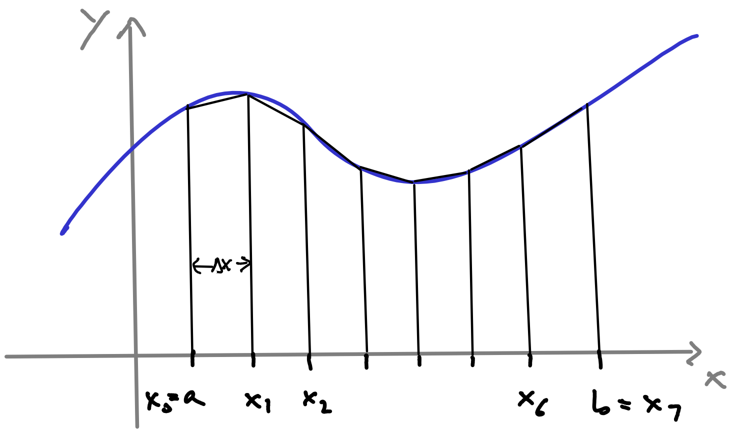
Goal: Find $\int_a^b f(x)\,dx$

Process: Subdivide $[a,b]$ into n equally long parts:

$$x_0 = a,\quad x_1 = a+\Delta x,\quad x_2 = a+2\Delta x,\ \ldots,\ x_n = a+n\Delta x = b$$

with $\Delta x = \frac{b-a}{n}$.

Then use trapezoids to estimate the area.



The ith trapezoid has area is $\frac{\Delta x}{2}\left(f(x_{i-1}) + f(x_i)\right)$.

So the total area is

$$\frac{\Delta x}{2}\sum_{i=1}^{n}\left(f(x_{i-1}) + f(x_i)\right) = \left(\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f(x_i)\right)\Delta x.$$

If you use twice as many trapzoids in the next step, then you can reuse the previous value as follows.

Let $T(n) = \frac{b-a}{n}\left(\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f\left(a+i\,\frac{b-a}{n}\right)\right)$.

Then $T(2n) = \frac{b-a}{2n}\left(\frac{f(a)+f(b)}{2} + \sum_{i=1}^{2n-1} f\left(a+i\,\frac{b-a}{2n}\right)\right)$ and

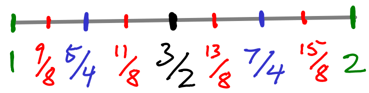
$$T(2n) = \frac{1}{2}T(n) + \frac{b-a}{2n}\sum_{i=1}^{n} f\left(a+(2i-1)\frac{b-a}{2n}\right).$$

Example: Approximate $\int_1^2 \sqrt{1+x}\,dx$ using 2, 4 and 8 intervals of the same length.

Sol^n: Start with one interval. Here $b=2$ and $a=1$

$$T(1) = \frac{1}{2}\left(f(2)+f(1)\right) = \frac{1}{2}\left(\sqrt{3}+\sqrt{2}\right) = 1.573132$$

$$T(2) = \frac{1}{2}T(1) + \frac{1}{2}f(3/2)$$



$$= \frac{1}{4}\left(\sqrt{3}+\sqrt{2}\right) + \frac{1}{2}\sqrt{\frac{13}{4}} = 1.577136$$

$$T(4) = \frac{1}{2}T(2) + \frac{1}{4}\left(f(5/4)+f(7/4)\right) = 1.578146$$

$$T(8) = \frac{1}{2}T(4) + \frac{1}{8}\left(f(9/8)+f(11/8)+f(13/8)+f(15/8)\right)$$

$$= 1.578399$$

The real value is

$$\int_1^2 \sqrt{1+x}\,dx = \left[\frac{2}{3}(1+x)^{3/2}\right]_1^2 = \frac{2}{3}\left(3^{3/2} - 2^{3/2}\right) = 1.578484$$

Taylor's Theorem

This theorem is about approximating an arbitrary continuous function by polynomials.

Problem: Find a polynomial $p(x) = ax^3 + bx^2 + cx + d$. Such that $p(0) = 3$, $p'(0) = -2$, $p''(0) = 1$ and $p'''(0) = 2$.

Sol^n: From $p(0) = d$ we get $d = 3$. Next

$p'(x) = 3ax^2 + 2bx + c$, so $p'(0) = c$ and $c = -2$. Next

$p''(x) = 6ax + 2b$ and $p''(0) = 2b = 1$.

So $b = \frac{1}{2}$. Finally, $p'''(x) = 6a$, so $p'''(0) = 2$ implies $a = \frac{2}{6}$

Altogether $p(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + 3$ is a polynomial satisfying the requirement.

Note that we can change $p(x)$ by adding terms with x^k for $k \geq 4$.

What if we want $p(2)$, $p'(2)$, etc. to be given values?

Sol^n: Write $p(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3$.

Then $p(2) = a$, $p'(2) = b$, $p''(2) = 2c$ and

$p'''(2) = 3!\, d$

In general: If $p(x) = \sum_{n=0}^{D} a_n (x-\lambda)^n$, then

$p^{(k)}(\lambda) = k!\, a_k$, where $p^{(k)}(x)$ is the k^{th} derivative of $p(x)$.

Taylor's Theorem: Let $f(x)$ be a function such that the k^{th} derivative of f exists for all $k \geq 0$ at a point λ. Then

$$f(x) = f(\lambda) + f'(\lambda)(x-\lambda) + \frac{f''(\lambda)}{2}(x-\lambda)^2 + \dots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(\lambda)}{k!}(x-\lambda)^k$$

for all x close to λ.

This series is called the Taylor series expansion of f at λ.

Example: Let $f(x) = e^x$, find its Taylor series at $\lambda = 0$.

Sol^n: Since $f'(x) = f(x)$ and $f(0) = 1$, we get

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

Fact: This is true for all $x \in \mathbb{R}$.

Often one does not want the whole series but simply take the first 2, 3 or 4 terms.

Some other Taylor series

Problem: Find the Taylor series for $\sin(x)$ at 0.

Sol^n: $f(x)=\sin(x)$, $f'(x)=\cos(x)$

$f''(x)=-\sin(x)$, $f'''(x)=-\cos(x)$

now it repeats. Since $\sin(0)=0$ and $\cos(0)=1$, we get

$$\sin(x) = 0 + x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + - \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}$$

Covered Chapters in Glyn James Book

Background Chapters 1 & 2.

Limits: Chapter 7.5, 7.8, 7.9

Differentiation: Chapter 8.1 – 8.5

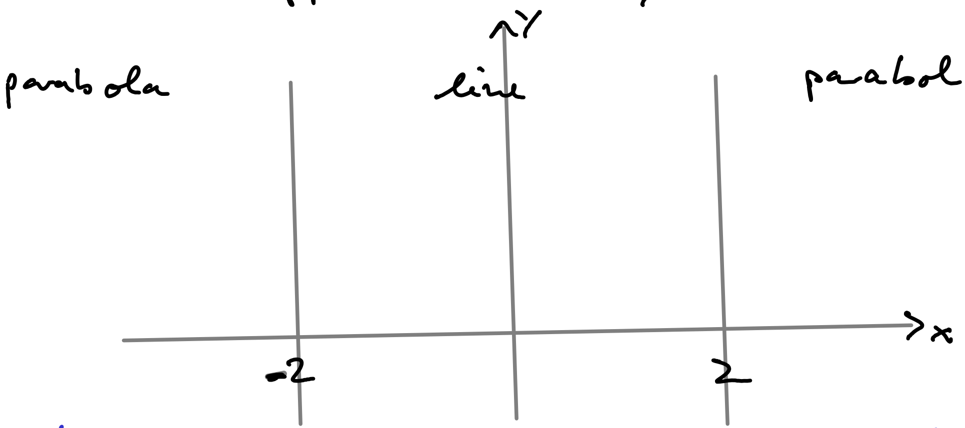
Integration: Chapter 8.7. – 8.10

Revision

Problem,

Let $f(x) = \begin{cases} 2x^2 - n, & x < -2 \\ mx + \ell; & -2 \le x \le 2 \\ kx^2 + 1, & 2 < x \end{cases}$

Find values for ℓ, m, n and k such that g is continuous and differentiable everywhere.



Since each piece is a polynomial, it is continuous. So to make g continuous, we need to match the pieces

$$\lim_{x \to -2^-} 2x^2 - n = \lim_{x \to -2^+} mx + \ell \quad \text{or} \quad 8 - n = -2m + \ell$$

$$\lim_{x \to 2^-} mx + \ell = \lim_{x \to 2^+} kx^2 + 1 \quad \text{or} \quad 2m + \ell = 4k + 1$$

In order to make g differentiable, we need the same slope at the meeting points.

$$\lim_{x \to -2^{-1}} 4x = m, \quad \text{i.e.} \quad -8 = m$$

$$m = \lim_{x \to 2^+} 2kx, \quad \text{i.e.} \quad m = 4k$$

So $m = -8$, $k = -2$, $\ell = 9$ and $n = -17$.

Problem: Evaluate $\lim_{x \to 0} \frac{x \sin(x)}{1 - \cos(x)}$. (Exam (2010/11) Q2(a))

Use l'Hospital's Rule:

$$\lim_{x \to 0} \frac{x \sin(x)}{1-\cos(x)} = \lim_{x \to 0} \frac{\sin(x) + x\cos(x)}{\sin(x)}$$

$$= \lim_{x \to 0} \frac{2\cos(x) - x\sin(x)}{\cos(x)} = \frac{2-0}{1} = 2.$$

Q2 b (Summer 2010/11)

Here $f(x) = x^4 - 8x^3 + 10$, so

$f'(x) = 4x^3 - 24x^2 = 4x^2(x-6)$ and

$f''(x) = 12x^2 - 48x = 12x(x-4)$

Critical points are where $f'(x) = 0$, i.e. $x = 0$ and $x = 6$.

Since $f''(0) = 0$, we have a point of inflection

From $f''(6) > 0$ we see that there is a local minimum at $x = 6$.

f is decreasing when $f' < 0$, i.e. $(-\infty, 6]$

Points of inflection are at 0 and 4

f is concave up if $f'' > 0$, $(-\infty, 0]$ and $[4, \infty)$

Now $f(0) = 10$, $f(4) = -246$ and $f(6) = -422$ and the graph looks as follows.