Q6(a)(ii) (Summer 2010/11)

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} \quad \text{when} \quad f(x)=\sqrt{x+1}$$

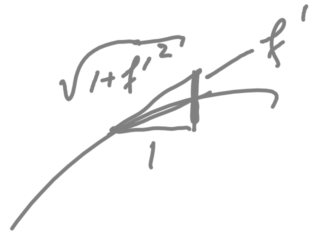
$$= \lim_{h\to 0}\frac{\sqrt{3+h}-\sqrt{3}}{h}\;\frac{\sqrt{3+h}+\sqrt{3}}{\sqrt{3+h}+\sqrt{3}}$$

$$= \lim_{h\to 0}\frac{\cancel{3}+h-\cancel{3}}{h(\sqrt{3+h}+\sqrt{3})} = \lim_{h\to 0}\frac{1}{\sqrt{3+h}+\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

Q7(c) Has a typo in 2010/11 and is correct in 2011/12.

Arc length of $\frac{2}{3}(x-1)^{3/2}$, $1\le x\le 4$.

$$\int_1^4 \sqrt{1+(x-1)}\,dx$$

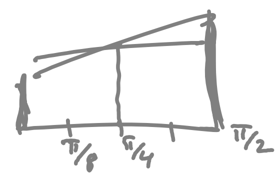


$(f'(x))^2$, where $f=\frac{2}{3}(x-1)^{3/2}$

$$= \int_1^4 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{16}{3}-\frac{2}{3}=\frac{14}{3}.$$

6(a) $f(x)=\sqrt{3+\sin^2(x)}$. Let $T(n)$ be the trapezoidal approximation of $2\int_0^{\pi/2} f(x)\,dx$ with n trapezoids.

$$T(1) = 2\,\frac{\pi}{2}\,\frac{f(0)+f(\pi/2)}{2} = \pi\,\frac{2+\sqrt{3}}{2}$$

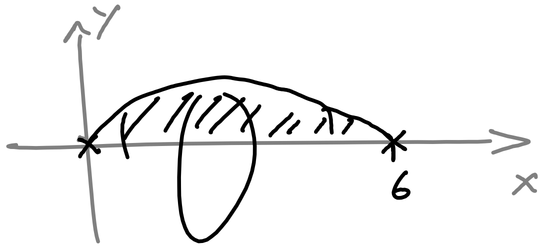


$$T(2) = \frac{1}{2}T(1) + \frac{\pi}{2}f(\pi/4) = 2\,\frac{\pi}{4}\left(\frac{f(0)+f(\pi/4)}{2}+\frac{f(\pi/4)+f(\pi/2)}{2}\right)$$

$$T(4) = \frac{1}{2} T(2) + \frac{\pi}{4} \left(f(\pi/8) + f(3\pi/8) \right)$$

7(b)

$$y = \frac{1}{9} x \sqrt{36 - x^2}$$



The volume is

$$V = \frac{\pi}{81} \int_0^6 x^2 (36 - x^2) \, dx$$

$$= \frac{\pi}{81} \left[12x^3 - \frac{1}{5} x^5 \right]_0^6$$

$$= \frac{\pi}{81} 6^4 \left(2 - \frac{6}{5} \right) = \frac{6^4 \pi}{81} \cdot \frac{4}{5}$$

$$= \frac{2^6}{5} \pi = \frac{64}{5} \pi = 12.8 \pi .$$

Summer 2011/12

Q1(c)

$$g(x) = \begin{cases} 3x + k, & x < 0 \\ e^{mx}, & x \geq 0 \end{cases}$$

continuity is only a problem at $x = 0$. From $\lim_{x \to 0^-} g(x) = k$ and $\lim_{x \to 0^+} g(x) = 1$, we see that $g(x)$ is continuous for $k = 1$

Similarly, differentiability is only an issue at $x = 0$.

$$g'(x) = \begin{cases} 3 & , \quad x < 0 \\ me^x & , \quad x \geq 0 \end{cases}$$

So we need $m = 3$ in order to enforce diffentiability.

Q4(b) $\frac{dy}{dt} = ky \rightsquigarrow \frac{1}{y}\frac{dy}{dt} = k \rightsquigarrow \int \frac{1}{y}\,dy = \int k\,dt$

$\ln(y) = kt + c \rightsquigarrow y = e^{kt+c} = e^c e^{kt}$

$$\boxed{y = Ae^{kt}}$$

half-life = time for 50% to decay, say $t_{1/2}$

$y(0) = A \qquad y(t_{1/2}) = \frac{1}{2}A = Ae^{kt_{1/2}}$

So $e^{kt_{1/2}} = \frac{1}{2}$ or $\ln(\frac{1}{2}) = kt_{1/2}$

or $k = \frac{-\ln(2)}{t_{1/2}} = \frac{-\ln(2)}{1200}$

Hence after 10 years we have

$y(10) = A\,e^{\frac{-\ln(2)}{1200} 10} = A\left(\frac{1}{2}\right)^{\frac{1}{120}} = 0.994A$

So 99.4% remain.