Q5(b) Summer Exam 2010/11

Fundamental Theorem of Calculus

If is continuous, then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

This implies

$$\frac{d}{dx}\int_{-x}^{x^3} \sqrt{t}\,\sin(t^2)\,dt = 3x^2\sqrt{x^3}\,\sin(x^6) - (-1)\sqrt{-x}\,\sin(x^2)$$

(b) (i) Use partial fractions

$$\frac{Ax+B}{x^2+2} + \frac{C}{x-2} = \frac{Ax^2-2Ax+Bx-2B+Cx^2+2C}{(x^2+2)(x-2)}$$

$$= \frac{(A+C)x^2+(B-2A)x+2C-2B}{(x^2+2)(x-2)}$$

$$\stackrel{!}{=} \frac{x^2-8x-6}{(x^2+2)(x-2)}$$

So $A+C=1$, $B-2A=-8$, $2C-2B=-6$

$$\Downarrow$$

$A+B=4 \Longleftarrow C=-3+B$

$B=4-A \implies 4-3A=-8$

$$\Downarrow$$

$B=0 \Longleftarrow A=4 \longrightarrow C=-3$

So $\displaystyle\int \frac{x^2-8x-6}{(x^2+2)(x-2)}\,dx = \int\left(\frac{4x}{x^2+2} - \frac{3}{x-2}\right)dx$

$\displaystyle = 2\int \frac{2x}{x^2+2}\,dx - 3\int \frac{1}{x-2}\,dx$

$= 2\ln(x^2+2) - 3\ln(x-2) + c.$

(b) $\displaystyle\int_1^e \sqrt{x}\,\ln(\sqrt{x})\,dx$ Try substitution $u=\sqrt{x}$

$\displaystyle\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$\displaystyle du = \frac{1}{2\sqrt{x}}\,dx$

$\displaystyle\int u \ln(u)\, 2u\,du$?? $\qquad dx = 2u\,du$

Try integration by parts.

$\displaystyle\int_1^e \underset{f'}{\sqrt{x}}\;\underset{g}{\ln(\sqrt{x})}\,dx$

$f = \frac{2}{3}x^{3/2}$

$g' = \frac{1}{\sqrt{x}}\frac{1}{2\sqrt{x}} = \frac{1}{2x}$

$\displaystyle = \left[\frac{2}{3}x^{3/2}\ln(\sqrt{x})\right]_1^e - \int_1^e \underbrace{\frac{2}{3}x^{3/2}\frac{1}{2x}}_{\frac{1}{3}x^{1/2}}\,dx$

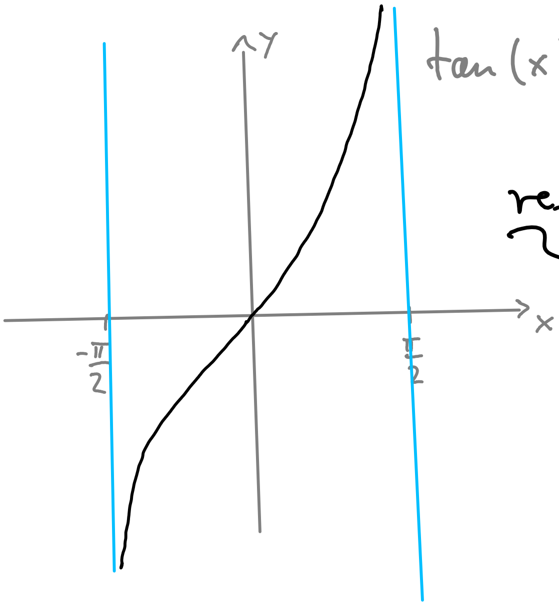
$\displaystyle = \frac{1}{3}e^{3/2} - \left[\frac{2}{9}x^{3/2}\right]_1^e = \frac{1}{3}e^{3/2} - \frac{2}{9}e^{3/2} + \frac{2}{9}$

$\displaystyle = \frac{1}{9}e^{3/2} + \frac{2}{9}.$

(iii) $\displaystyle\int_0^\infty \frac{x}{1+x^4}\,dx$ Substitute $u = x^2$

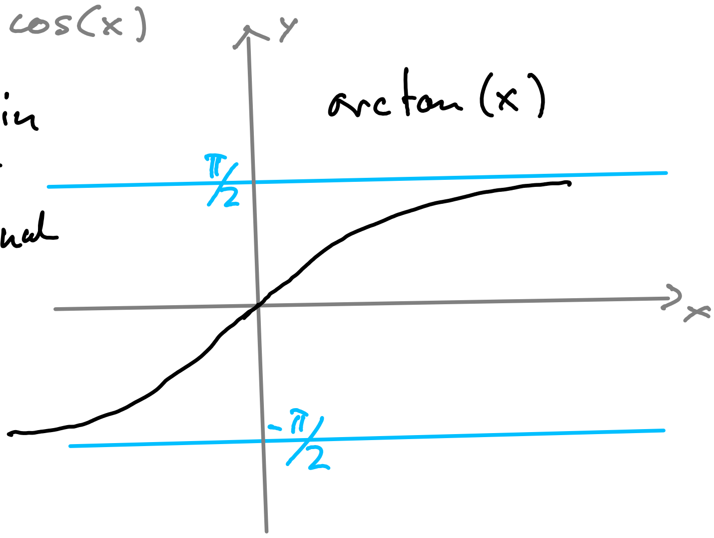
$du = 2x\,dx$

$$= \frac{1}{2}\int_0^\infty \frac{1}{1+u^2}\,du = \left[\frac{1}{2}\arctan(u)\right]_0^\infty = \frac{1}{2}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{4}.$$

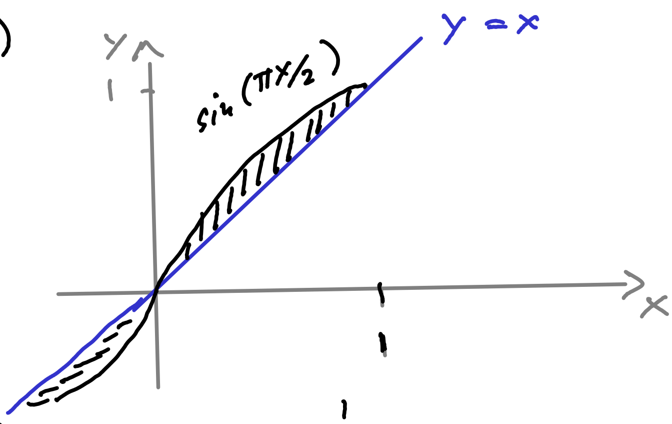


$\tan(x) = \dfrac{\sin(x)}{\cos(x)}$

reflect in main diagonal



7(a)



The area is $\displaystyle 2\int_0^1 \left(\sin\left(\frac{\pi x}{2}\right) - x\right)dx$

$$= 2\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{1}{2}x^2\right]_0^1$$

$$= 2\left(-\frac{1}{2} - \left(-\frac{2}{\pi}\right)\right) = \frac{4}{\pi} - 1$$