Taylor's Theorem

This theorem is about approximating an arbitrary continuous function by polynomials.

Problem: Find a polynomial $p(x) = ax^3 + bx^2 + cx + d$.
Such that $p(0) = 3$, $p'(0) = -2$, $p''(0) = 1$ and $p'''(0) = 2$.

Sol^n: From $p(0) = d$ we get $d = 3$. Next

$p'(x) = 3ax^2 + 2bx + c$, so $p'(0) = c$ and $c = -2$. Next

$p''(x) = 6ax + 2b$ and $p''(0) = 2b = 1$.

So $b = \frac{1}{2}$. Finally, $p'''(x) = 6a$, so $p'''(0) = 2$ implies $a = \frac{2}{6}$

Altogether $p(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + 3$ is a polynomial satisfying the requirement.

Note that we can change $p(x)$ by adding terms with x^k for $k \geq 4$.

What if we want $p(2)$, $p'(2)$, etc. to be given values?

Sol^n: Write $p(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3$.

Then $p(2) = a$, $p'(2) = b$, $p''(2) = 2c$ and

$p'''(2) = 3!\,d$

In general: If $p(x) = \sum_{n=0}^{D} a_n (x-\lambda)^n$, then

$$p^{(k)}(\lambda) = k!\, a_k,$$ where $p^{(k)}(x)$ is the k^{th} derivative of $p(x)$.

Taylor's Theorem: Let $f(x)$ be a function such that the k^{th} derivative of f exists for all $k \geq 0$ at a point λ. Then

$$f(x) = f(\lambda) + f'(\lambda)(x-\lambda) + \frac{f''(\lambda)}{2}(x-\lambda)^2 + \dots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(\lambda)}{k!}(x-\lambda)^k$$

for all x close to λ.

This series is called the Taylor series expansion of f at λ.

Example: let $f(x) = e^x$, find its Taylor series at $\lambda = 0$.

Solⁿ: Since $f'(x) = f(x)$ and $f(0) = 1$, we get

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

Fact: This is true for all $x \in \mathbb{R}$.

Often one does not want the whole series but simply take the first 2, 3 or 4 terms.