Numerical Integration

Recall the very first attempts at integration, using small rectangles to approximate area.

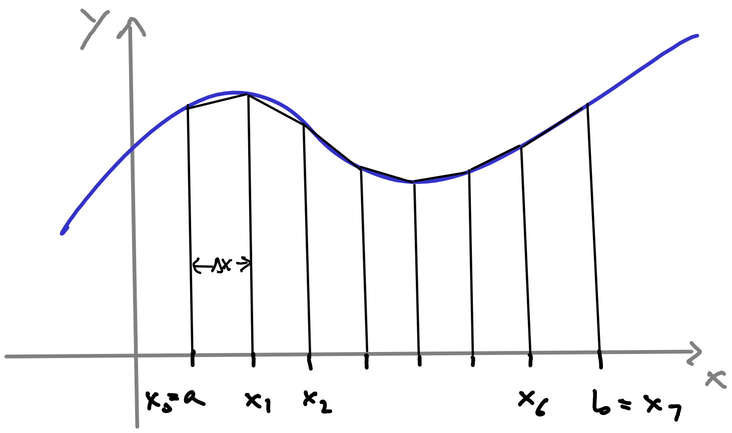
<u>Goal:</u> Find $\int_a^b f(x)\,dx$

<u>Process:</u> Subdivide $[a,b]$ into n equally long parts:

$$x_0 = a \;,\; x_1 = a + \Delta x \;,\; x_2 = a + 2\Delta x \;, \ldots, \; x_n = a + n\Delta x = b$$

with $\Delta x = \frac{b-a}{n}$.

Then use trapezoids to estimate the area.



The ith trapezoid has area is $\frac{\Delta x}{2}\Big(f(x_{i-1}) + f(x_i) \Big)$.

So the total area is

$$\frac{\Delta x}{2} \sum_{i=1}^{n} \Big(f(x_{i-1}) + f(x_i) \Big) = \left(\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f(x_i) \right) \Delta x .$$

If you use twice as many trapzoids in the next step, then you can reuse the previous value as follows.

Let $T(n) = \frac{b-a}{n}\left(\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f\left(a+i\frac{b-a}{n}\right)\right)$.

Then $T(2n) = \frac{b-a}{2n}\left(\frac{f(a)+f(b)}{2} + \sum_{i=1}^{2n-1} f\left(a+i\frac{b-a}{2n}\right)\right)$ and

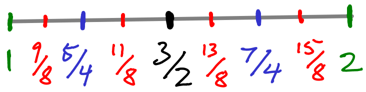
$$T(2n) = \frac{1}{2}T(n) + \frac{b-a}{2n}\sum_{i=1}^{n} f\left(a+(2i-1)\frac{b-a}{2n}\right).$$

<u>Example</u>: Approximate $\int_1^2 \sqrt{1+x}\,dx$ using 2, 4 and 8 intervals of the same length.

<u>Sol^n</u>: Start with one interval. Here $b=2$ and $a=1$

$$T(1) = \frac{1}{2}\left(f(2)+f(1)\right) = \frac{1}{2}\left(\sqrt{3}+\sqrt{2}\right) = 1.573132$$

$$T(2) = \frac{1}{2}T(1) + \frac{1}{2}f(3/2)$$



$$= \frac{1}{4}\left(\sqrt{3}+\sqrt{2}\right) + \frac{1}{2}\sqrt{\frac{13}{4}} = 1.577136$$

$$T(4) = \frac{1}{2}T(2) + \frac{1}{4}\left(f(5/4)+f(7/4)\right) = 1.578146$$

$$T(8) = \frac{1}{2}T(4) + \frac{1}{8}\left(f(9/8)+f(11/8)+f(13/8)+f(15/8)\right)$$

$$= 1.578399$$

The real value is

$$\int_1^2 \sqrt{1+x}\,dx = \left[\frac{2}{3}(1+x)^{3/2}\right]_1^2 = \frac{2}{3}\left(3^{3/2} - 2^{3/2}\right) = 1.578484$$