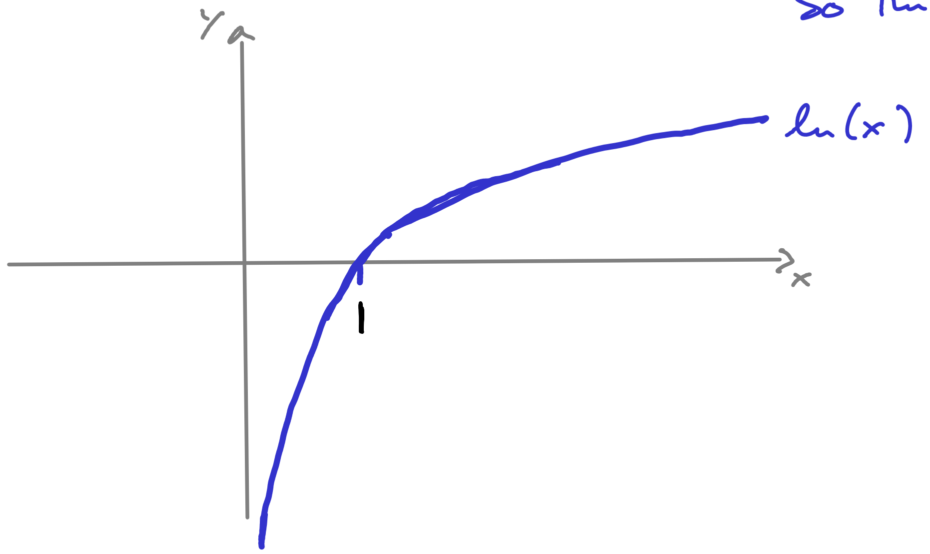
# Some more improper integrals

**Problem:** Does $\displaystyle\int_0^1 \frac{1}{x}\,dx$ exist?

**Sol^n:** $\displaystyle\int_\varepsilon^1 \frac{1}{x}\,dx = \Big[\ln(x)\Big]_\varepsilon^1 = 0 - \ln(\varepsilon) \xrightarrow{\varepsilon \to 0} \infty$

So this integral does not exist.




$\displaystyle\lim_{x\to\infty} \ln(x) = \infty$

**Problem:** Does $\displaystyle\int_2^\infty \frac{x^2-1}{x^3-3x+4}\,dx$ exist?

Substitution: $u = x^3 - 3x + 4$. So $\dfrac{du}{dx} = 3x^2 - 3 = 3(x^2-1)$

or $\dfrac{du}{3} = (x^2-1)\,dx$ and we get

$$\int \frac{x^2-1}{x^3-3x+4}\,dx = \int \frac{1}{3}\,\frac{1}{u}\,du = \frac{1}{3}\ln(u)$$

$$= \frac{1}{3}\ln\left(x^3-3x+4\right)$$

So $\displaystyle\int_2^\infty \frac{x^2-1}{x^3-3x+4}\,dx = \left[\frac{1}{3}\ln\left(x^3-3x+4\right)\right]_2^\infty = \frac{1}{3}\lim_{L\to\infty} \ln\left(L^3-3L+4\right) - \frac{1}{3}\ln(6)$

Again, this integral does not exist.

Problem: What about $\displaystyle\int_2^\infty \frac{x}{x^3-3x-2}\,dx$ ?

Roughly speaking, for large $x$ we integrate $\frac{1}{x^2}$, so we expect the integral to exist.

Partial fractions:

1. Factorise $x^3-3x-2$.

   $-1$ is a zero and we find

$$\begin{array}{l} (x^3-3x-2)/(x+1) = x^2-x-2 \\ \underline{x^3+x^2} \\ \quad -x^2-3x-2 \\ \quad \underline{-x^2-x} \\ \qquad -2x-2 \end{array}$$

$$x = \frac{1}{2} \pm \sqrt{\frac{1}{4}+2} = \begin{cases} 2 \\ -1 \end{cases}$$

So $x^3-3x-2 = (x+1)^2(x-2)$.

Now solve $\dfrac{Ax+B}{(x+1)^2} + \dfrac{C}{x-2} = \dfrac{x}{x^3-3x-2}$.

$$(Ax+B)(x-2) + C(x+1)^2 = x$$

Put $x=2$ to get $9C=2$, i.e. $C=\frac{2}{9}$

Put $x=-1$ to get $-3(B-A) = -1$, i.e. $B-A = \frac{1}{3}$

Put $x=0$ to get $-2B+C=0$, so $2B=\frac{2}{9}$ and $B=\frac{1}{9}$, and hence $A = \frac{1}{9}-\frac{1}{3} = -\frac{2}{9}$

Check: $\frac{1}{9}\left(\frac{-2x+1}{(x+1)^2} + \frac{2}{x-2}\right)$

$$= \frac{1}{9} \frac{\cancel{-2x^2} + 5x \cancel{-2} + 2(\cancel{x^2} + 2x \cancel{+1})}{(x+1)^2(x-2)}$$

$$= \frac{1}{9} \frac{9x}{(x+1)^2(x-2)} = \frac{x}{(x+1)^2(x-2)}$$

So $\displaystyle\int_2^\infty \frac{x}{x^3-3x-2}\,dx = \frac{1}{9}\int_2^\infty \left(\frac{-2x+1}{(x+1)^2} + \frac{2}{x-2}\right) dx$

$$= \frac{1}{9}\int_2^\infty \frac{-2x+1}{(x+1)^2}\,dx + \frac{2}{9}\int_2^\infty \frac{1}{x-2}\,dx$$

$$= \frac{1}{9}\int_2^\infty \frac{-2x-2+3}{(x+1)^2}\,dx + \frac{2}{9}\Big[\ln(x-2)\Big]_2^\infty$$

$$= \quad \cdots \quad + \frac{2}{9}\left(\underbrace{\lim_{L\to\infty}(L-2)}_{\infty} - \underbrace{\lim_{\varepsilon\to 2^+}(\varepsilon-2)}_{-\infty}\right)$$

And we cannot decide this way!