Calculating Arc Length

Example: Find the length of the curve $y = x^2 - \frac{1}{8}\ln(x)$ for $1 \le x \le 2$.

Sol^n: The length is

$$\int_1^2 \sqrt{1 + \left(2x - \frac{1}{8x}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}}\, dx$$

$$= \int_1^2 \sqrt{4x^2 + \frac{1}{2} + \frac{1}{64x^2}}\, dx = \int_1^2 \sqrt{\left(2x + \frac{1}{8x}\right)^2}\, dx$$

$$= \int_1^2 \left(2x + \frac{1}{8x}\right) dx = \left[x^2 + \frac{1}{8}\ln(x)\right]_1^2 = 4 + \frac{1}{8}\ln(2) - (1+0)$$

$$= 3 + \frac{1}{8}\ln(2)$$

Example: Find the length of $y = \frac{2}{3}(x-1)^{3/2}$ for $1 \le x \le 4$.

Sol^n: $y' = (x-1)^{1/2}$, so the length is

$$\int_1^4 \sqrt{1 + (x-1)}\, dx = \int_1^4 \sqrt{x}\, dx = \left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$$

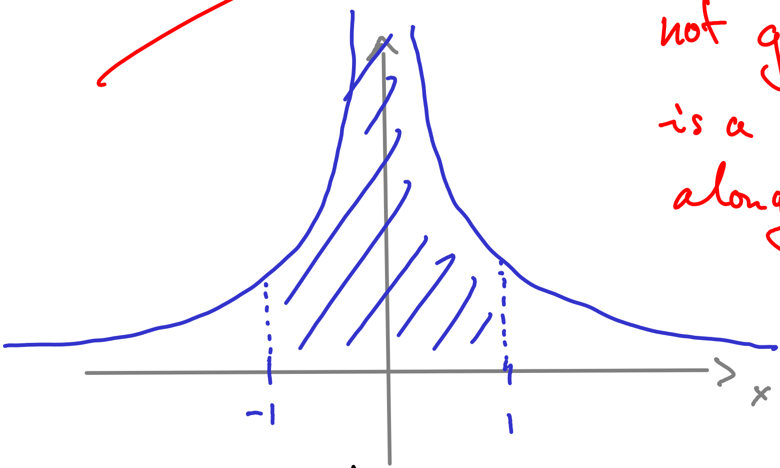
Improper Integrals

An integral in which one or both boundaries are infinite or discontinuities of the function to be integrated are called improper. They may or may not exist.

Problem: Find the are under the graph of $f(x) = \frac{1}{x^2}$ between $-1 \le x \le 1$.

Fine, we calculate $\cancel{\int_{-1}^{1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-1}^{1} = -1-1=-2.}$

not good! There is a discontinuity along the way.



Solution: Take $\int_{\varepsilon}^{1} \frac{1}{x^2}\,dx$ then let $\underset{\varepsilon>0}{\varepsilon \to 0}$.

$$\int_{\varepsilon}^{1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{\varepsilon}^{1} = -1 + \frac{1}{\varepsilon} \xrightarrow{\varepsilon \to 0} \infty$$

Conclusion: Even half the area is not finite!

Another example: What about the area under $y = \frac{1}{x^2}$ between 1 and ∞?

Idea: Calculate $\int_1^L \frac{1}{x^2}\,dx$ and then let $L \to \infty$.

$$\int_1^L \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^L = -\frac{1}{L} + 1 \xrightarrow{L\to\infty} 1$$

Convention: We write $\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^b f(x)\,dx$

or even $\int_{-\infty}^\infty f(x)\,dx$ in order to shorten

$$\lim_{L\to\infty} \int_a^L f(x)\,dx, \quad \lim_{L\to -\infty} \int_{-\infty}^b f(x)\,dx \quad \text{or} \quad \lim_{L\to\infty} \int_{-L}^L f(x)\,dx.$$

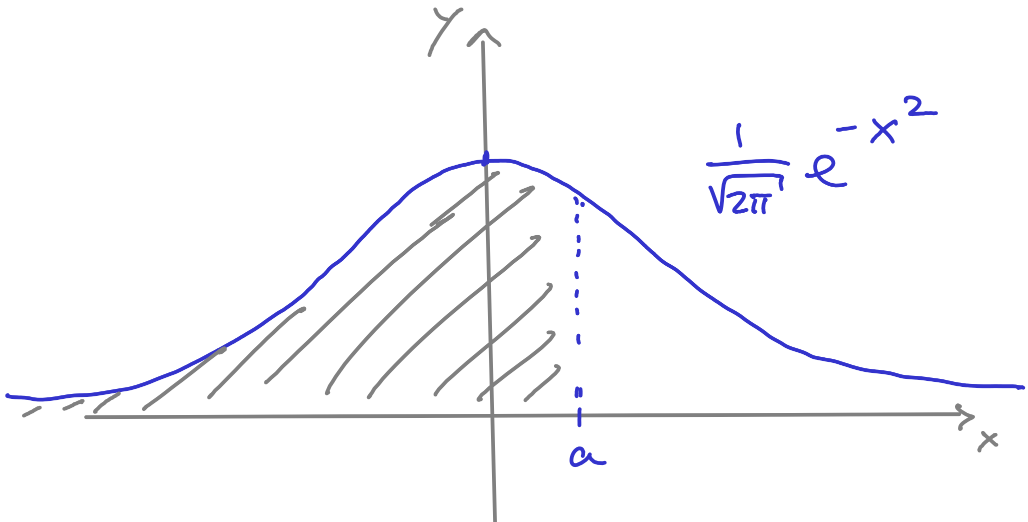
Note: Being a limit, these integrals may not exist.

Example: $\int_1^\infty \frac{1}{x}\,dx = \left[\ln(x)\right]_1^\infty = \lim_{L\to\infty} \ln(L) - 0$

does not exist.

Example:
$$\int_{-\infty}^{0} e^x \, dx = \left[e^x \right]_{-\infty}^{0} = 1 - \lim_{L \to -\infty} e^L = 1$$

In probability a density function $f : \mathbb{R} \to \mathbb{R}$ is a function with $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1$.



$$P(x \leq a) = \int_{-\infty}^{a} f(x)\, dx$$