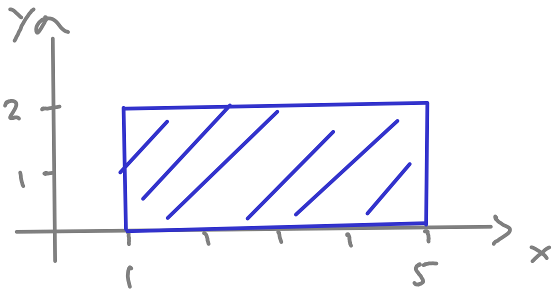
Examples of centroids

Problem: Find the centre of mass of the rectangle
$[1,5]\times[0,2]$



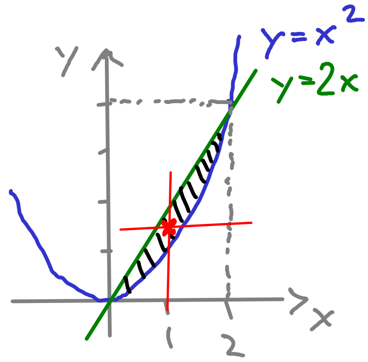
Sol^n: The area is $A=8$.

$$\bar{x} = \frac{1}{A}\int_1^5 2x\,dx = \frac{1}{8}\left[x^2\right]_1^5 = \frac{1}{8}(25-1) = 3$$

$$\bar{y} = \frac{1}{2A}\int_1^5 4\,dx = \frac{1}{16}\left[4x\right]_1^5 = \frac{1}{16}(20-4) = 1.$$

Problem: Find the centre of mass of the bounded area between the curves $y=x^2$ and $y=2x$.

Sol^n:



① The area is

$$A = \int_0^2 (2x - x^2)\,dx = \left[x^2 - \tfrac{1}{3}x^3\right]_0^2$$

$$= 4 - \frac{8}{3} = \frac{4}{3}$$

② $$\bar{x} = \frac{1}{A}\int_0^2 x(2x - x^2)\,dx = \frac{3}{4}\left[\tfrac{2}{3}x^3 - \tfrac{1}{4}x^4\right]_0^2$$

$$= \frac{3}{4}\left(\frac{16}{3} - 4\right) = 1$$

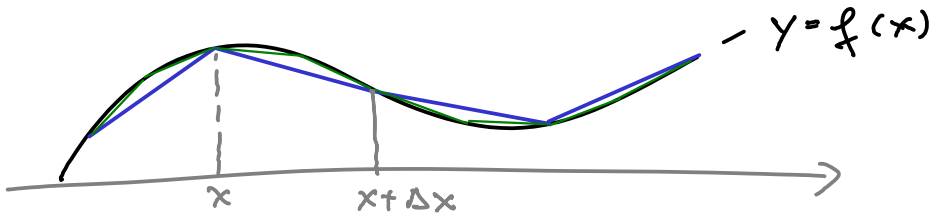
$$\bar{y} = \frac{1}{2A}\int_0^2 \left((2x)^2 - x^4\right)dx = \frac{3}{8}\left[\tfrac{4}{3}x^3 - \tfrac{1}{5}x^5\right]_0^2$$

$$= \frac{3}{8}\left(32\left(\frac{1}{3} - \frac{1}{5}\right)\right) = \frac{24}{15} = \frac{8}{5}$$

Lengths of Curves

How long is the graph of the sine wave from 0 to π?

Or a piece of a parabola?



<u>Idea</u>: Approximate the curve by short line segments, say from $(x, f(x))$ to $(x+\Delta x, f(x+\Delta x))$, sum them up and then take the limit when $\Delta x \to 0$, i.e. the number of pieces goes to infinity

The length of the line segment from $(x, f(x))$ to $(x+\Delta x, f(x+\Delta x))$ is

$$\sqrt{(\Delta x)^2 + (f(x+\Delta x) - f(x))^2}$$

$$= \sqrt{(\Delta x)^2 \left(1 + \left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right)^2\right)}$$

$$= \Delta x \sqrt{1 + \left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right)^2} \xrightarrow[\Delta x \to 0]{} \Delta x \sqrt{1 + [f'(x)]^2}$$

Summing all the lengths of the pieces and taking the limit results in the formula for the length:

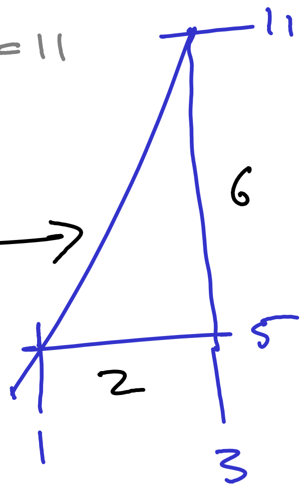
$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Easy example: Find the length of $f(x) = 3x + 2$ between $x = 1$ and $x = 3$.

Sol^n: $L = \int_1^3 \sqrt{1 + 3^2}\, dx = \sqrt{10}\,[x]_1^3 = 2\sqrt{10}$

Check: $f(1) = 5$, $f(3) = 11$

$\sqrt{40} = 2\sqrt{10}$ $\longrightarrow$



Warning: Often the integral one gets is near impossible to work out!

One example you might try is the cycloid, the curve described by a point on a wheel rolling along