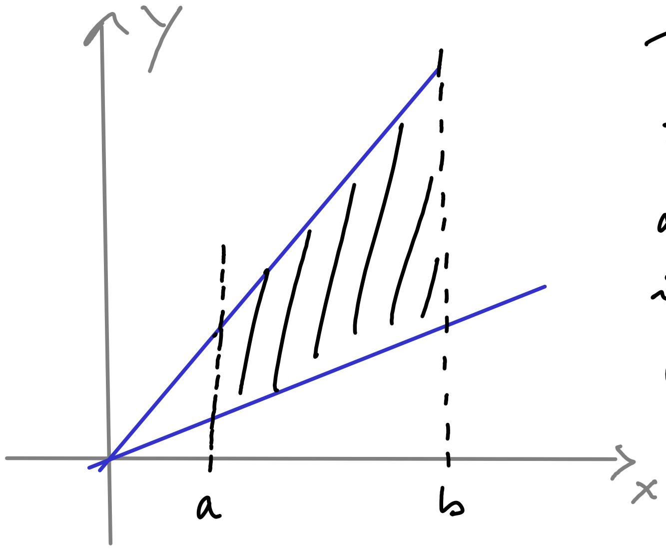
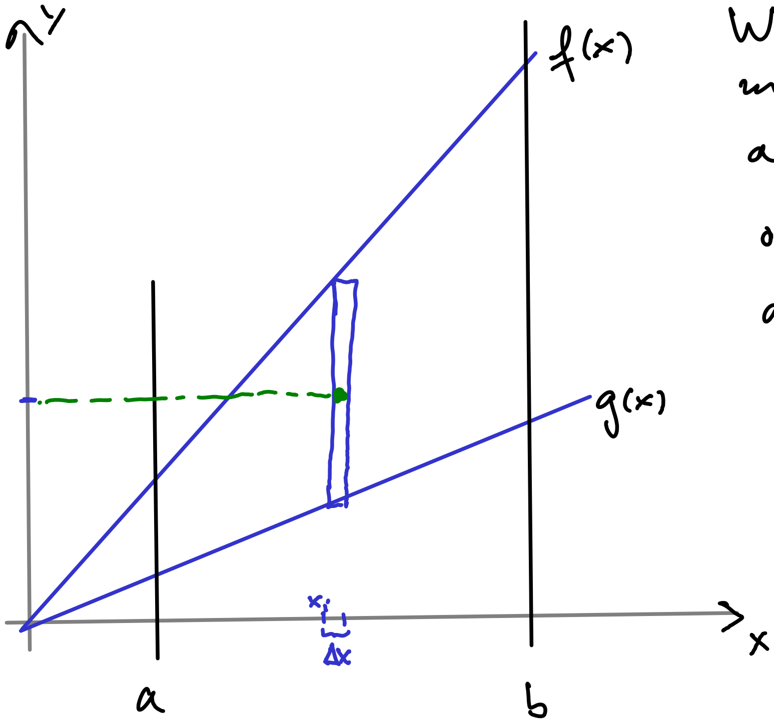
Centroids aka centre of mass



The centre of mass of an 2D object is that point at which you can ballance it on a pencil.
(True for convex objects.)

More precisely it is the point about which no momentum is generated by the object.



We'll calcalate the momentum about each axis, assuming the object has homogeneous density ρ.

Moment about the x-axis: 1 For a thin strip:

$$\underbrace{\frac{g(x_i)+f(x_i)}{2}}_{\text{distance of mid point from x-axis}} \underbrace{\left(f(x_i)-g(x_i)\right)\Delta x\, \rho}_{\text{mass}}$$

② Summing these for n strips and taking the limit when $n \to \infty$ we get

$$\frac{\rho}{2}\int_a^b (f(x))^2 - (g(x))^2 \, dx$$

Moment about the y-axis: ① For the same strip

$$x_i \underbrace{(f(x_i) - g(x_i))\,\Delta x\,\rho}_{\text{mass}}$$

distance from y-axis

② Again taking the limit, we get

$$\rho \int_a^b x\,(f(x) - g(x))\,dx$$

The whole object has mass

$$M = \rho \int_a^b (f(x) - g(x))\,dx \qquad (\text{area} \times \text{density} = \text{mass})$$

The coordinates of the centre of mass, say $\bar{x}$ and $\bar{y}$, must satisfy:

$\bar{x}M$ = moment about y-axis

$\bar{y}M$ = moment about x-axis

Hence

$$\bar{x} = \frac{\rho}{M}\int_a^b (f(x) - g(x))\,x\,dx$$

$$\bar{y} = \frac{\rho}{2M}\int_a^b \left((f(x))^2 - (g(x))^2\right) dx$$

We can cancel ϱ here to get

$$\bar{x} = \frac{1}{A} \int_a^b x\,(f(x) - g(x))\,dx$$

and

$$\bar{y} = \frac{1}{2A} \int_a^b (f(x)^2 - g(x)^2)\,dx,$$

where A is the area of the object:

$$A = \int_a^b (f(x) - g(x))\,dx.$$