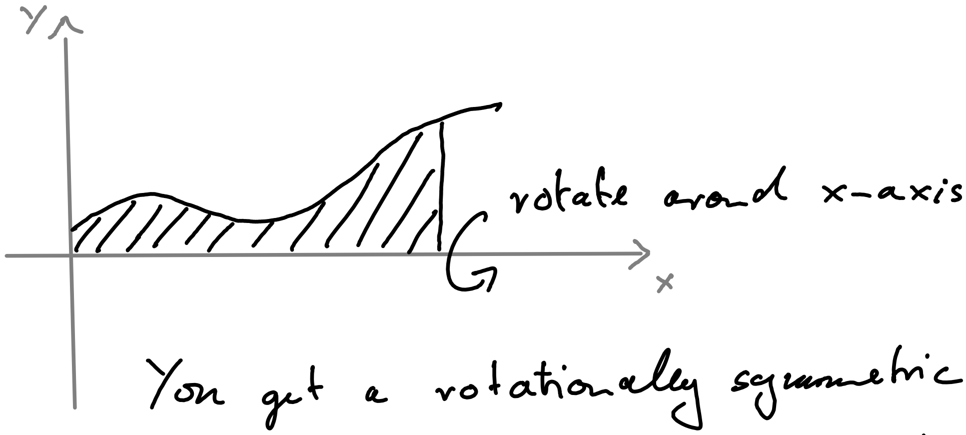
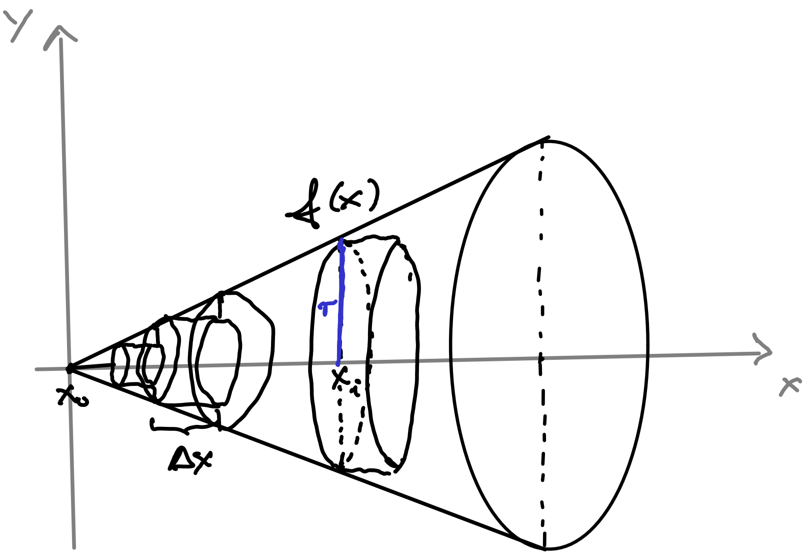
Applications of Integration

Volumes of revolution



You get a rotationally symmetric object in 3D and one can ask for its volume.

The idea is to slice the object into approximately discs of thickness Δx and then the volume is the sum of the volumes of those disks and we take the limit $\Delta x \to 0$.



Each disc has volume $\pi[f(x_i)]^2 \Delta x$ where $x_i = x_0 + i\,\Delta x$.

So the volume is approximated by

$$V \approx \sum_{i=0}^{n-1} \Delta x\, \pi \left[f(x_i) \right]^2$$ if we have n discs.

In the limit we get

$$V = \int_a^b \pi [f(x)]^2 \, dx$$, where a and b are the left and right boundaries.

Example: Calculate the volume of the cone of height 6 generated by revolving $f(x) = \frac{1}{2}x$ about the x-axis.

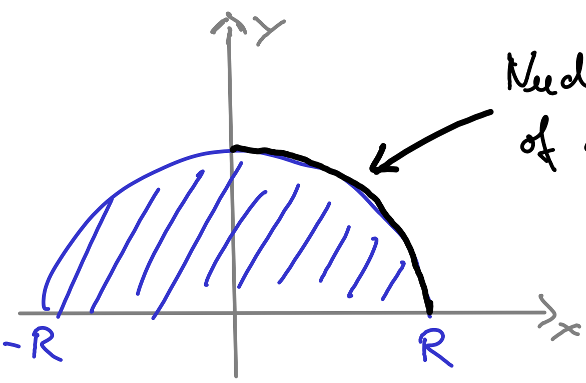
Sol^n

$$\pi \int_0^6 \left(\frac{1}{2}x\right)^2 dx = \pi \int_0^6 \frac{1}{4}x^2 \, dx = \pi \left[\frac{1}{12}x^3\right]_0^6$$

$$= 18\pi$$

Example: Calculate the volume of a sphere of radius R.

Sol^n: Rotate the half disc of radius R about the x-axis.



Need this as the graph of a function f(x)

Maybe be lazy and only calculate half the volume by only considering the quarter disc with $x \geq 0$.

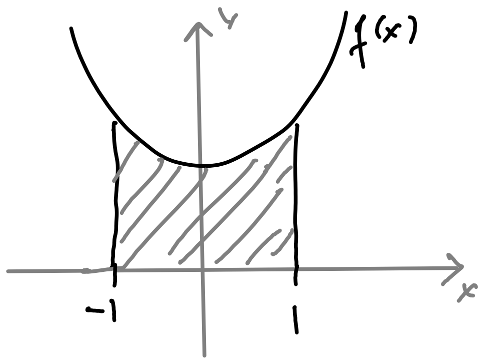
$x^2 + y^2 = R^2$ has as solutions the points (x, y) on the circle.

So $y = f(x) = \sqrt{R^2 - x^2}$ and the volume of the sphere

is $$V = 2\pi \int_0^R (R^2 - x^2)\, dx = 2\pi \left[R^2 x - \frac{1}{3} x^3 \right]_0^R$$

$$= 2\pi \left(R^3 - \frac{1}{3} R^3 \right) = \frac{4}{3} \pi R^3$$

<u>Last example</u>: Find the volume of the solid of revolution obtained by rotating the are under $f(x) = x^2 + 2$ between $x = -1$ and $x = 1$ about the x-axis



$$V = \pi \int_{-1}^{1} (x^2+2)^2\, dx = \pi \int_{-1}^{1} (x^4 + 4x^2 + 4)\, dx$$

$$= \pi \left[\frac{1}{5} x^5 + \frac{4}{3} x^3 + 4x \right]_{-1}^{1} = \pi \left(\frac{1}{5} + \frac{4}{3} + 4 - \left(-\frac{1}{5} - \frac{4}{3} - 4 \right) \right)$$

$$= 2\pi \left[\frac{3 + 20 + 60}{15} \right] = \frac{166}{15} \pi = 11\pi + \frac{1}{15}\pi$$