More integration

Problem: Find $\int \cos^2(x)\,dx$.

Sol^n: Use the addition theorem for cosine to get

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$

Then
$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$$
$$= 2\cos^2(\alpha) - (\cos^2(\alpha) + \sin^2(\alpha))$$
$$= 2\cos^2(\alpha) - 1$$

So $\cos^2(\alpha) = \frac{1}{2}(\cos(2\alpha)+1)$, and hence

$$\int \cos^2(x)\,dx = \frac{1}{2}\int (\cos(2x)+1)\,dx$$
$$= \frac{1}{4}\sin(2x) + \frac{1}{2}x + c$$

Problem: Find $\int \underbrace{\sin(x)}_{f}\,\underbrace{\cos(x)}_{g'}\,dx$

$\int fg'\,dx = fg - \int f'g\,dx$

Use integration by parts:

$$\int \sin(x)\cos(x)\,dx = \sin^2(x) - \int \cos(x)\sin(x)\,dx$$

$$2\int \sin(x)\cos(x)\,dx = \sin^2(x)$$

$$\int \sin(x)\cos(x)\,dx = \frac{1}{2}\sin^2(x) + c$$

Substitution

Recall the chain rule:

$$\frac{d}{dx}\Big(f(g(x)) \Big) = f'(g(x))\, g'(x)$$

Integrating both sides gives

$$f(g(x)) = \int f'(g(x))\, g'(x)\, dx$$

$$= \int f'(g)\, dg \qquad\qquad \frac{dg}{dx} = g'(x)$$

How is it used?

Let's do $\int \cos(x) \sin(x)\, dx$ again.

Put $g = \sin(x)$. Then $dg = \cos(x)\, dx$ and

$$\int \sin(x) \cos(x)\, dx = \int g\, dg = \frac{1}{2} g^2 + c$$

$$= \frac{1}{2} \sin^2(x) + c.$$

Another example: Find $\int x e^{x^2+2}\, dx$

Put $g = x^2+2$. Then $dg = 2x\, dx$ and

$$\int x e^{x^2+2}\, dx = \int \frac{1}{2} e^g\, dg = \frac{1}{2} e^g + c = \frac{1}{2} e^{x^2+2} + c.$$

Problem: Find $\int \frac{1}{a^2+x^2}\,dx$, where $a>0$.

Put $x = a\tan(u)$. Then $\frac{dx}{du} = \frac{d}{du}\left(\frac{\sin(u)}{\cos(u)}\right)$

$u = \arctan\left(\frac{x}{a}\right)$

$$= \frac{\cos^2(u)+\sin^2(u)}{\cos^2(u)} = \frac{1}{\cos^2(u)}$$

So $dx = \frac{a}{\cos^2(u)}\,du$ and

$$\int \frac{1}{a^2+x^2}\,dx = \int \frac{1}{a^2+a^2\tan^2(u)} \frac{a}{\cos^2(u)}\,du$$

$$= \int \frac{1}{a^2\left(1+\frac{\sin^2(u)}{\cos^2(u)}\right)} \frac{a}{\cos^2(u)}\,du$$

$$= \frac{1}{a}\int \frac{1}{\frac{\cos^2(u)+\sin^2(u)}{\cos^2(u)}} \cdot \frac{1}{\cos^2(u)}\,du$$

$$= \frac{1}{a}\int du = \frac{1}{a}u + c$$

$$= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c.$$

One more: Find $\int \sqrt{1-x^2}\,dx$

Put $x = \sin(u)$. Then $u = \arcsin(x)$ and

$dx = \cos(u)\,du$ and

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2(u)}\,\cos(u)\,du = \int \cos^2(u)\,du$$

$$= \frac{1}{4}\sin(2u) + \frac{1}{2}u + c = \frac{1}{4}\sin(2\arcsin(x)) + \frac{1}{2}\arcsin(x) + c.$$