# More on partial fractions

## Fundamental Theorem of Algebra

Let $p(x)$ be a polynomial of degree at least one. Then

(i) There is a root/zero of $p(x)$ in $\mathbb{C}$, the complex numbers.

(ii) If $a$ is a root of $p(x)$, i.e. $p(a)=0$, then $p(x) = (x-a)\,q(x)$, where $q(x)$ is a polynomial of degree one less than $p$.

Consequence: Over $\mathbb{C}$, every polynomial is a product of linear factors, that is of factors of the form $(x-a)$, where $a=$ const.

Note: This is not true over $\mathbb{R}$, because some quadratic polynomials without roots.

Example: $p(x) = x^2+1$ has no real root.

In the context of partial fractions, this means we need to deal with the case where the denominator has such an irreducible quadratic factor.

**Example:** Write $\dfrac{1}{(x-1)(x^2+2)}$ as partial fraction.

**Sol$^n$:** Want $$\frac{1}{(x-1)(x^2+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2}$$

Now $$\frac{A}{x-1} + \frac{Bx+C}{x^2+2} = \frac{A(x^2+2) + (Bx+C)(x-1)}{(x-1)(x^2+2)}$$

$$= \frac{(A+B)x^2 + (C-B)x + 2A-C}{(x-1)(x^2+2)}$$

So $A+B=0$, $C-B=0$, $2A-C=1$, and

$B=-A$, $B=C$, and then $3A=1$.

**Hence:** $$\frac{1}{(x-1)(x^2+1)} = \frac{1}{3}\frac{1}{x-1} - \frac{1}{3}\frac{x+1}{x^2+2}$$

$$= \frac{1}{3}\left(\frac{1}{x-1} - \frac{x+1}{x^2+2}\right)$$

$$= \frac{1}{3} \frac{x^2+2 - (x+1)(x-1)}{(x-1)(x^2+2)}$$

Now we can calculate

$$\int \frac{1}{(x-1)(x^2+1)}\,dx = \frac{1}{3}\int\left(\frac{1}{x-1} - \frac{x+1}{x^2+2}\right)dx$$

$$= \frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{x}{x^2+2}\,dx - \frac{1}{3}\int \frac{1}{x^2+2}\,dx$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\int \frac{2x}{x^2+2}\,dx - \frac{1}{3}\int \frac{1}{x^2+2}\,dx$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+2) - \frac{1}{3}\int \frac{1}{x^2+2}\,dx$$

From the tables we find

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) \qquad (a>0).$$

So

$$\int \frac{1}{x^2+2}\,dx = \int \frac{1}{x^2+\sqrt{2}^2}\,dx = \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)$$

and

$$\int \frac{1}{(x-1)(x^2+2)}\,dx = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+2) - \frac{1}{3\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + c$$

## Long division with polynomials

Factorising polynomials of (high?) degree can be done by iterating

① Find a zero, say $a$, of $p(x)$.

② Find $q(x)$ s.t. $p(x) = q(x)(x-a)$

Example: Factorise $p(x) = x^3-2x^2-x+2$.

Trial and error shows that $x=1$ is a root.

$$(x^3-2x^2-x+2)/(x-1) = x^2-x-2$$

$$\begin{array}{l} x^3-x^2 \\ \hline -x^2-x+2 \\ -x^2+x \\ \hline -2x+2 \\ -2x+2 \\ \hline 0 \end{array}$$

So $x^3-2x^2-x+2 = (x-1)(x^2-x-2)$

$$(x^2-x-2)/(x-2) = x+1$$

$$\begin{array}{l} x^2-2x \\ \hline x-2 \end{array}$$

Now $x^3-2x^2+x+2 = (x-1)(x-2)(x+1)$.