Techniques for Integration

Using the chain rule we find

$$\frac{d}{dx}\left(\ln\left(f(x)\right)\right) = \frac{1}{f(x)}\,\frac{df}{dx}(x) = \frac{f'(x)}{f(x)}.$$

So we find that

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left(f(x)\right) + c$$

Example: Find $\displaystyle\int \frac{6x^2-8x+2}{x^3-2x^2+x+3}\,dx$.

Sol^n: $\displaystyle\int \frac{6x^2-8x+2}{x^3-2x^2+x+3}\,dx = 2\int \frac{3x^2-4x+1}{x^3-2x^2+x+3}\,dx$

$$= 2\ln\left(x^3-2x^2+x+3\right) + c$$

Problem: Find $\displaystyle\int \frac{3}{x^2+2x-3}\,dx$.

Sol^n: $\displaystyle\frac{3}{x^2+2x-3} = \frac{3}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$

$$= \frac{A(x+3)}{(x-1)(x+3)} + \frac{B(x-1)}{(x-1)(x+3)}$$

$$= \frac{(A+B)x + 3A - B}{(x-1)(x+3)}$$

Since we want both sides to be equal, we need $(A+B)x + 3A - B = 3$, which means

$$A+B=0 \quad \text{and} \quad 3A-B=3$$

$$B=-A \longrightarrow 4A=3$$

$$B=-3/4 \longleftarrow A=3/4$$

Now we win:

$$\int \frac{3}{x^2+2x-3}\,dx = \int \frac{3/4}{x-1} - \frac{3/4}{x+3}\,dx$$

$$= 3/4 \int \frac{1}{x-1}\,dx - 3/4 \int \frac{1}{x+3}\,dx$$

$$= 3/4 \ln|x-1| - 3/4 \ln|x+3| + c$$

$$= 3/4 \left(\ln|x-1| - \ln|x+3| \right) + c$$

$$= 3/4 \ln\left(\frac{|x-1|}{|x+3|} \right) + c$$

This trick is known as the partial fractions method.

Integration by parts

Using the product rule (for differentiation), we find

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}.$$

Now integrate both sides:

$$fg = \int\left(g\frac{df}{dx} + f\frac{dg}{dx}\right)dx$$

$$= \int g\frac{df}{dx}\,dx + \int f\frac{dg}{dx}\,dx$$

$$= \int g f'\,dx + \int f g'\,dx$$

From this we get that

$$\int f g'\,dx = fg - \int f' g\,dx.$$

Example: Find $\int x \ln(x)\,dx$.

Sol^n: let $f(x) = \ln(x)$ and $g'(x) = x$. Then

$f'(x) = \frac{1}{x}$ and $g(x) = \frac{1}{2}x^2$ and

$$\int x \ln(x) = \frac{1}{2}x^2 \ln(x) - \int \frac{1}{x}\,\frac{1}{2}x^2\,dx$$

$$= \frac{1}{2}x^2 \ln(x) - \frac{1}{2}\int x\,dx$$

$$= \frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + c$$

Check: $\frac{d}{dx}\left(\frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + c\right)$

$$= x\ln(x) + \cancel{\frac{1}{2}x^2\frac{1}{x}} - \cancel{\frac{1}{2}x} = x\ln(x).$$

Problem: Find $\int x e^x\,dx$.

Sol^n: Put $f(x) = x$ and $g'(x) = e^x$

Then $f'(x) = 1$ and $g(x) = e^x$.

So $\int x e^x\,dx = x e^x - \int e^x\,dx = e^x(x-1)$