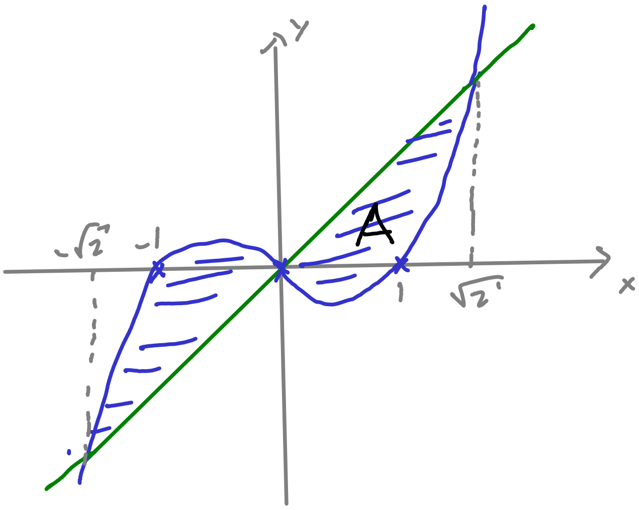
Applications of integrals

Problem: Calculate the bounded area between the graph of $f(x) = x^3 - x$ and the line $y = x$.



$f = x^3 - x = x(x^2-1) = x(x+1)(x-1)$

$f' = 3x^2 - 1$

Points of intersection:

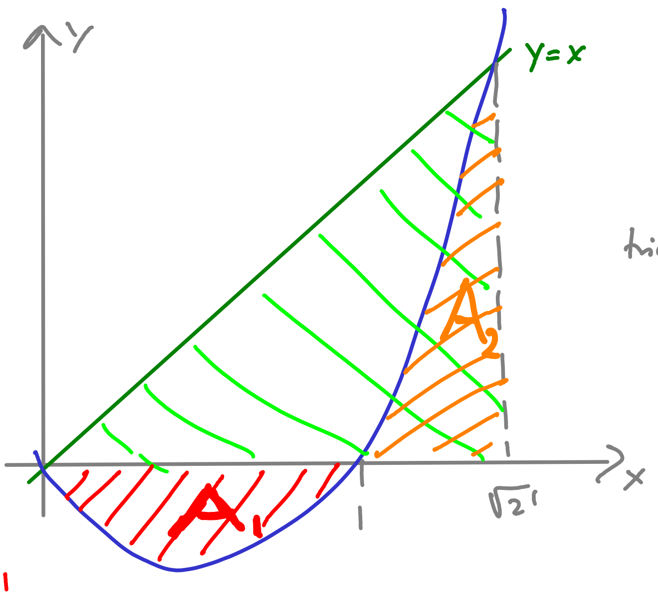
$f(x)$ and $g(x)$ intersect when $f(x) = g(x)$.

Here $x^3 - x = x$ or $x^3 - 2x = 0$

or $x(x^2 - 2) = 0$, so $x = 0$ or $x = \pm\sqrt{2}$

Hence the area is twice (by symmetry) the area between the graphs and $x=0$ and $x=\sqrt{2}$.

Remember, area is only given by an integral if $f(x) \geq 0$



We want

$T - A_2 + A_1$

triangle

$$A_2 = \int_1^{\sqrt{2}} (x^3 - x)\,dx$$

$$= \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_1^{\sqrt{2}}$$

$$= 0 - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{4}$$

$$A_1 = -\int_0^1 x^3 - x\,dx = -\left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_0^1$$

$$= -\left(\frac{1}{4} - 0\right) = \frac{1}{4}$$

The triangle has are $\frac{1}{2}\sqrt{2}^2 = 1$, which also is the are of the bonded piece, ad the are in question is twice this, i.e. 2.

Alternatively: Notice that for $0 \le x \le \sqrt{2}$, the graph of $y = x$ is always above the graph of $y = x^3 - x$. This means that $x \ge x^3 - x$ for these values of x. In other words, $2x - x^3 \ge 0$ and the area of on piece can be calculated as

$$\int_0^{\sqrt{2}} (2x - x^3)\, dx = \left[x^2 - \frac{1}{4}x^4 \right]_0^{\sqrt{2}} = 2 - 1 - 0 = 1.$$

In general: The are between two consecutive intersections of $f(x)$ and $g(x)$, at $x=a$ and $x=b$ say, is given by

$$\int_a^b f(x) - g(x)\, dx \qquad \text{if} \quad f(x) \ge g(x)$$

or by $$\int_a^b g(x) - f(x)\, dx \qquad \text{if} \quad g(x) \ge f(x).$$

Remark: If we interpret $\int_a^b f(x)\,dx$ as area, then area below the x-axis is counted negatively.