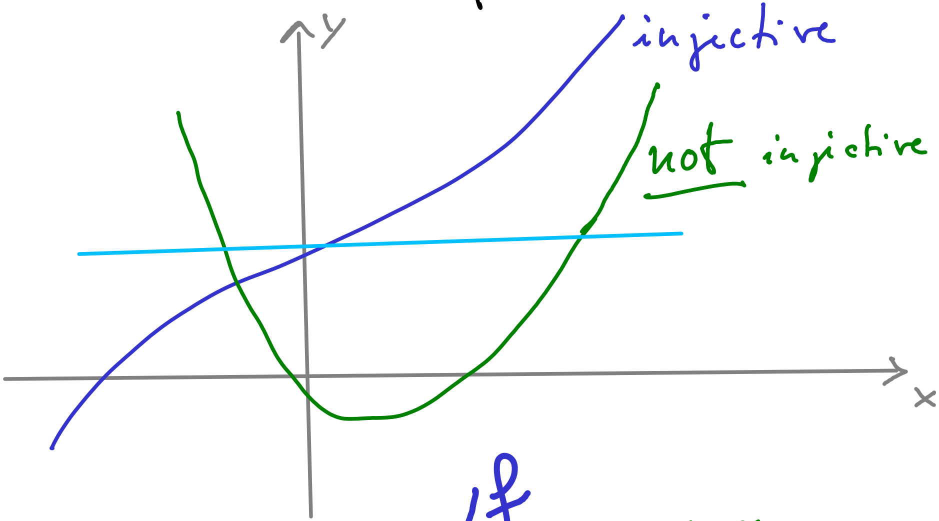
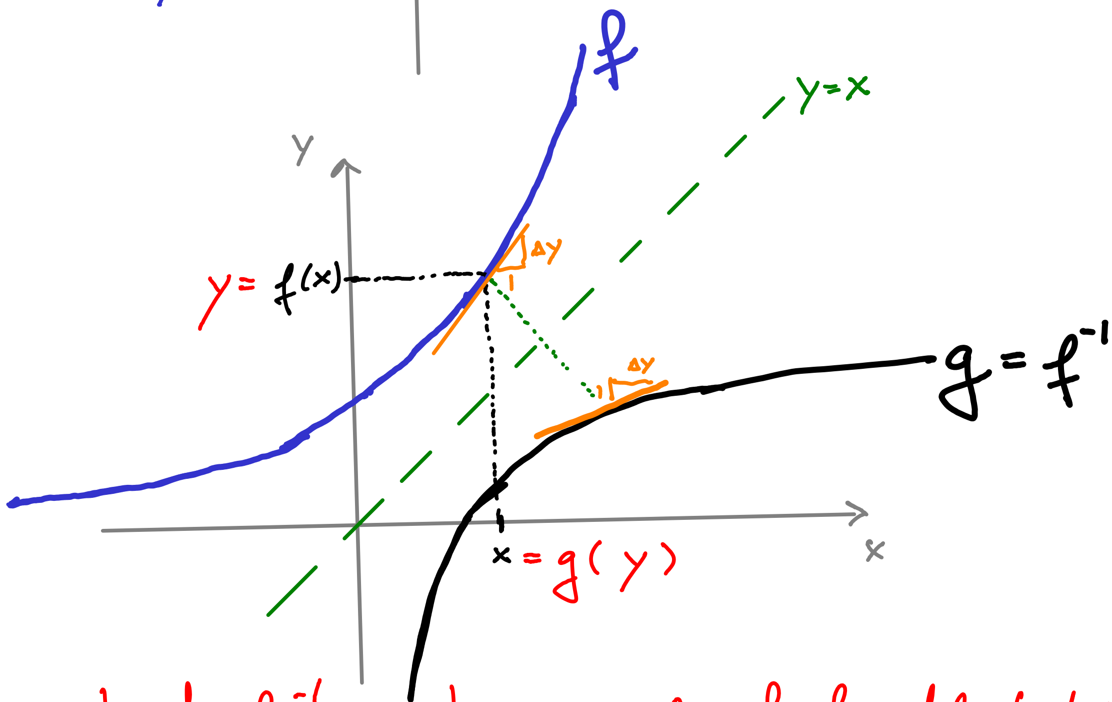
Inverse Functions

Definition: A function $f : [a,b] \to \mathbb{R}$, where $a, b \in \mathbb{R}$, $a < b$, is <u>invertible</u> if there exists a function g s.t. $g(f(x)) = x$ for all $x \in [a,b]$.

Note, for such a g to exist, f must satisfy the condition that $f(x_1) = f(x_2)$ implies $x_1 = x_2$, in which case f is called <u>injective</u>.

Another way of saying this is that every horizontal line meets the graph of f in at most one point.





The graph of f^{-1} is the graph of f reflected in $y=x$.

Derivative of inverse function

Fact: f has an inverse if f is injective.

Suppose f has an inverse, say f^{-1}, then

$f^{-1}(f(x)) = x$ and using the chain rule,

we get

$$(f^{-1})'(f(x))\, f'(x) = 1$$

Similarly, $f(f^{-1}(x)) = x$ and

$$f'(f^{-1}(x))\,(f^{-1})'(x) = 1$$

which says

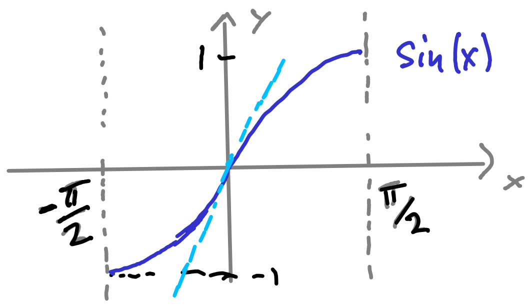
$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

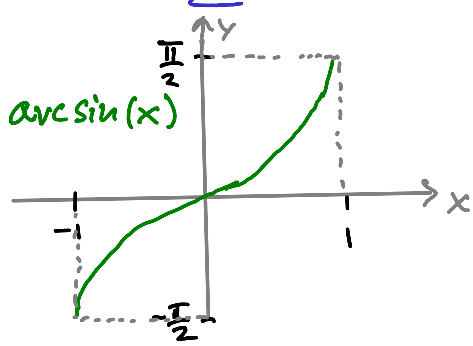
Example: ① Let $f(x) = e^x$, then $f^{-1}(x) = \ln(x)$

So $\frac{d}{dx}(\ln(x)) = \frac{1}{f'(\ln(x))} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$

which we already knew.

② Let $f(x) = \sin(x)$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$





Problem: Find $\frac{d}{dx}\left(\arcsin(x)\right)$.

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\cos\left(\arcsin(x)\right)}$$