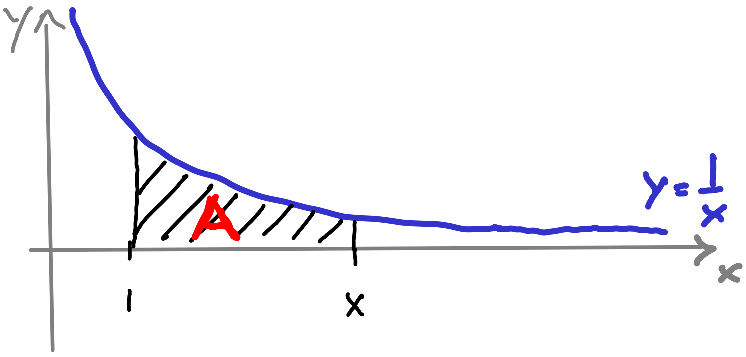
Logarithm defined properly

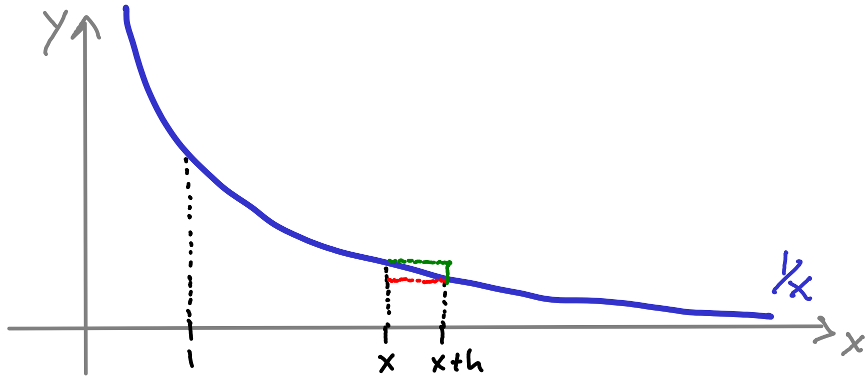
Definition: For $x > 0$ let A be the are shown below.



Define $\ln(x) = \begin{cases} A & x \geq 1 \\ -A & x < 0 \end{cases}$

Theorem: For $x > 0$, we have $\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}$.

Proof:



$$\frac{d}{dx}\left(\ln(x)\right) = \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h}$$

$$\frac{h}{x+h} < \ln(x+h) - \ln(x) < \frac{h}{x}$$

Dividing by h gives

$$\frac{1}{x+h} < \frac{\ln(x+h) - \ln(x)}{h} < \frac{1}{x}$$

So the limit is squeezed between $\lim_{h \to 0} \frac{1}{x+h} = \frac{1}{x}$ and $\frac{1}{x}$

and so $\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}$.

Consequences:

① For $x, y > 0$, then $\ln(xy) = \ln(x) + \ln(y)$.

The reason is $\frac{d}{dx}(\ln(xy)) = \frac{y}{xy} = \frac{1}{x} = \frac{d}{dx}(\ln(x))$

So $\frac{d}{dx}(\ln(xy) - \ln(x)) = 0$ and hence

$\ln(xy) - \ln(x) = c$, a constant.

With $x = 1$, we get $\ln(y) = c$, which then

gives $\ln(xy) - \ln(x) = \ln(y)$, or $\ln(xy) = \ln(x) + \ln(y)$.

② One can also show $\ln(x^k) = k\ln(x)$

Application: Find the derivative of

$$f(x) = \frac{(x+1)^2(x-2)^3(x+4)}{(x-1)^2(x-6)}$$

Idea: Use $\ln$ to turn the product into a sum.

$$\ln(f(x)) = \ln\left((x+1)^2(x-2)^3(x+4)(x-1)^{-2}(x-6)^{-1}\right)$$

$$= \ln((x+1)^2) + \ln((x-2)^3) + \ln(x+4) + \ln((x-1)^{-2}) + \ln((x-6)^{-1})$$

$$= 2\ln(x+1) + 3\ln(x-2) + \ln(x+4) - 2\ln(x-1) - \ln(x-6)$$

Next take derivative

$$\frac{f'(x)}{f(x)} = \frac{2}{x+1} + \frac{3}{x-2} + \frac{1}{x+4} - \frac{2}{x-1} - \frac{1}{x-6}$$

So $f'(x) = f(x)\left[\frac{2}{x+1} + \frac{3}{x-2} + \frac{1}{x+4} - \frac{2}{x-1} - \frac{1}{x-6}\right]$.

$$= \frac{(x+1)^2(x-2)^3(x+4)}{(x-1)^2(x-6)}\left[\frac{2}{x+1} + \frac{3}{x-2} + \frac{1}{x+4} - \frac{2}{x-1} - \frac{1}{x-6}\right]$$

A little bit on differential equations

A differential equation, is an equation involving a variable and its derivative.

The simplest example is

$$\frac{dy}{dx} = ky \quad , \quad k \text{ a const. and } y = y(x).$$

Are there any solutions? YES: $y = e^{kx}$

More general: $y = Ce^{kx}$, C a const.

This is a solution, because

$$\frac{dy}{dx} = Cke^{kx} = ky.$$

Are these all the solutions?

Suppose $y(x)$ and $z(x)$ are both solutions.

Then $\frac{d}{dx}\left(\frac{y}{z}\right) = \frac{y'z - z'y}{z^2} = \frac{kyz - kzy}{z^2} = 0$,

So $\frac{y}{z} = K = \text{const.}$, i.e $y = Kz$, that is any two solutions are multiples of each other.