Logarithms and exponents

Laws of exponents

* $(a^b)^c = a^{bc}$
* $a^b a^c = a^{b+c}$
* If $a > 0$, then $a^b > 0$ for all b.
* $a^0 = 1$
* If $n \in \mathbb{N}$, then $a^{1/n} = \sqrt[n]{a}$.
* $a^{-b} = \frac{1}{a^b}$

If $b \in \mathbb{N}$, then
$a^b = \underbrace{a\,a \cdots a}_{b \text{ many } a\text{'s}}$.

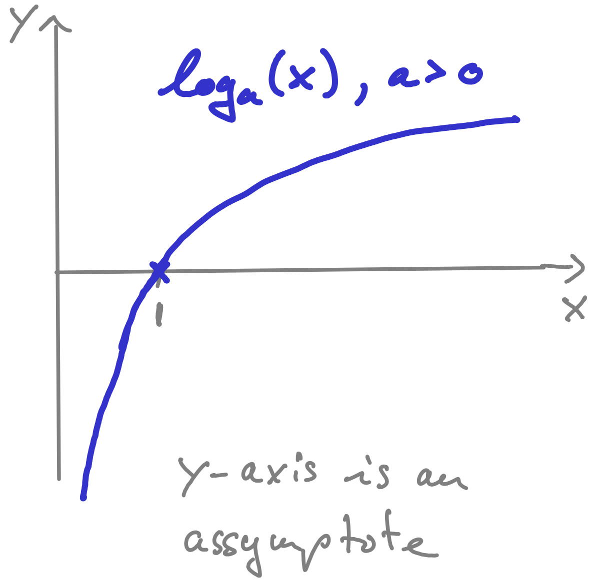
Logarithms are the inverses of exponentiation.

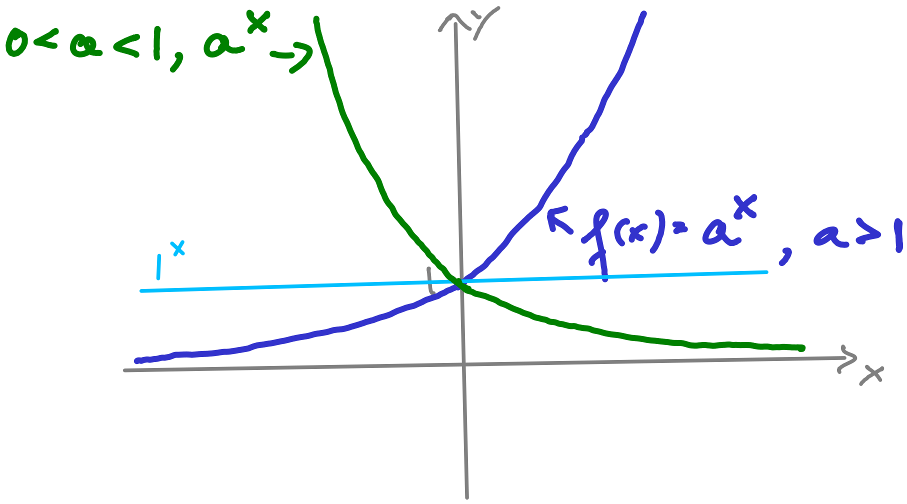
$2^4 = 16$	$\log_2(16) = 4$
$4^3 = 64$	$\log_4(64) = 3$
$4^{1/2} = 2$	$\log_4(2) = \frac{1}{2}$
$8 = 2^3 = (4^{1/2})^3 = 4^{3/2}$	$\log_4(8) = \frac{3}{2}$

Rules for logarithms

* $\log_a(bc) = \log_a(b) + \log_a(c)$
* $\log_a(b^c) = c \log_a(b)$

* $\log_a(1) = 0$
* For $a > 0$, $\log_a : \mathbb{R}_{>0} \to \mathbb{R}$
* $a^{\log_a(b)} = b$
* $\lim_{x \to \infty} \log_a(x) = \infty$





$\lim_{x \to -\infty} a^x = 0$, $a > 1$

Recall: Our favourite base for exponentiation and logarithms is e (Euler's number)

This is because $\frac{d}{dx}(e^x) = e^x$

Now what is $\frac{d}{dx}(a^x)$?

$$a^x = \left(e^{\log_e(a)}\right)^x = e^{x \log_e(a)}$$

So $\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{x \log_e(a)}\right) = \log_e(a)\, e^{x \log_e a} = \log_e(x) a^x$

Convention: Instead of $\log_e$ we write ln or log.

One can use logarithm to solve equations where the target occurs in the exponent

Example: Solve $2^{x^2-2x+1} = 64$.

Sol^n: Apply $\log_2$ on both sides to get

$$\log_2\left(2^{x^2-2x+1}\right) = \log_2(64)$$

or $\quad x^2-2x+1 = 6$

or $\quad x^2-2x-5 = 0$

So $\quad x = 1 \pm \sqrt{6}$

A proper definition of $\ln(x)$ involves an integral.

Recall that $\frac{d}{dx}(x^n) = n\,x^{n-1}$, $n \in \mathbb{R}$

In particular, for $n \in \mathbb{Z}$, we get x^{n-1} as the derivative of $\frac{1}{n}x^n$. Except for $n=0$!

So x^{-1} does never occur as derivative here.

Fact: $\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x} = x^{-1}$

So the proper definition of $\ln(x)$ is as the area under the graph of $f(x) = \frac{1}{x}$ and above the x-axis.