Optimisation Problems

Problem: You have the choice of a rectangular shaped garden of 60 square metres. What side lengths should the garden have in order to minimise the length of the fence around it?

Sol^n: Let a and b be the side lengths.

Then area $= ab = 60$ and perimeter is

$$P = 2a + 2b = \frac{120}{b} + 2b = \frac{2b^2 + 120}{b}.$$

By the quotient rule $\left[\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \right]$

$$\frac{dP}{db} = \frac{4bb - (2b^2 + 120)}{b^2} = \frac{2b^2 - 120}{b^2}.$$

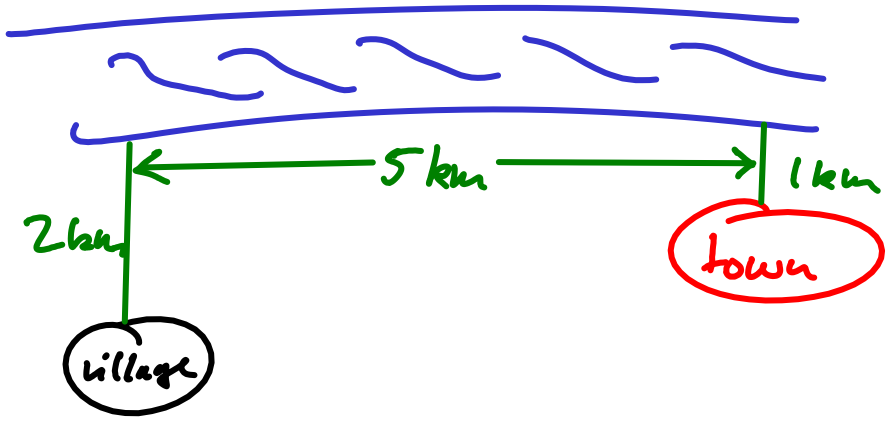
Now $\frac{dP}{db} = 0$ if and only if $2b^2 - 120 = 0$

or $b^2 = 60$, i.e. $b = \pm\sqrt{60}$.

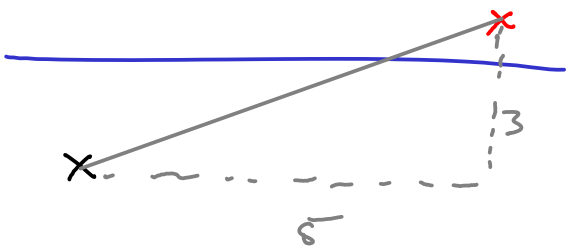
Since $-\sqrt{60}$ makes no sense, the garden should be a square of side length $\sqrt{60}$.

Problem: A farmer needs to go into town but needs to wash his potatoes in the river on the way. Both, his village is 2km and the town 1km away from the river shore. The town and village are 5km appart in the direction of the river.

Find the shortest path.



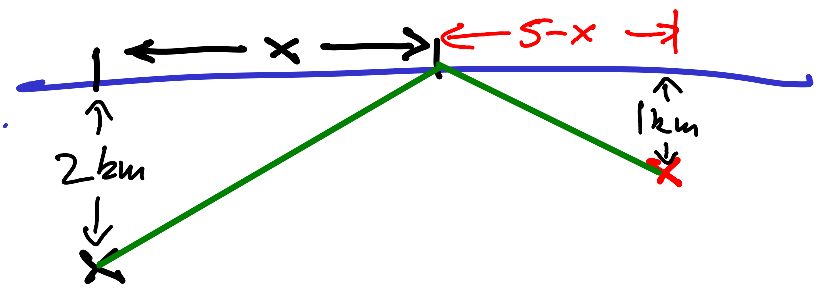
Sol^n



graphical

$\sqrt{34}$

Using functions: let $f(x)$ be the length of the path, when you meet the river at x



$$f(x) = \sqrt{4 + x^2} + \sqrt{1 + (5-x)^2}$$

$$\frac{df}{dx} = \frac{x}{\sqrt{4+x^2}} + \frac{-(5-x)}{\sqrt{1+(5-x)^2}}$$

$$= \frac{x\sqrt{1+(5-x)^2} + (x-5)\sqrt{4+x^2}}{\sqrt{4+x^2}\,\sqrt{1+(5-x)^2}}$$

$\frac{df}{dx} = 0$ when $x\sqrt{1+(5-x)^2} + (x-5)\sqrt{4+x^2} = 0$

or $x\sqrt{1+(5-x)^2} = (5-x)\sqrt{4+x^2}$

or $x^2(1+(5-x)^2) = (5-x)^2(4+x^2)$

or $x^2 + \cancel{x^2(5-x)^2} = \cancel{x^2(5-x)^2} + 4(5-x)^2$

or $0 = 4(25-10x+x^2) - x^2$

$= 3x^2 - 40x + 100$

or $0 = x^2 - \frac{40}{3}x + \frac{100}{3}$

$x = \frac{20}{3} \pm \sqrt{\frac{400}{9} - \frac{300}{9}}$

$= \frac{20}{3} \pm \frac{10}{3} = \begin{cases} 10 \\ \frac{10}{3} \end{cases}$

The problem suggests $x = \frac{10}{3}$.

Then $f\left(\frac{10}{3}\right) = \sqrt{4 + \frac{100}{9}} + \sqrt{1 + \left(5 - \frac{10}{3}\right)^2}$

$= \sqrt{\frac{136}{9}} + \sqrt{\frac{34}{9}} = \sqrt{4 \cdot \frac{34}{9}} + \sqrt{\frac{34}{9}}$

$= \sqrt{34}$