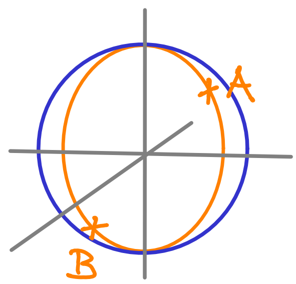
Intermediate Value Theorem:

Suppose that f is continuous at all points x with $a \le x \le b$. Assume also that $f(a)f(b) \le 0$. Then there is a c with $a \le c \le b$ such that $f(c) = 0$.

We already used it when we showed that $x^3 + 1 = 3x$ has three real solutions.

Another example:

Take any great circle on the earth.



Fact: There are diametrically opposite points A and B at which the air pressure (or temperature) is equal.

The idea is that the function

$$f(\theta) = \text{pressure at } A - \text{pressure at } B$$

is continuous. Here θ varies from 0 to π.

Also note that $f(0) = -f(\pi)$

So either $f(0) = f(\pi) = 0$ and we found the two points, or $f(0)$ and $f(\pi)$ have opposite signs and by the theorem $f(\theta) = 0$ for some θ in between 0 and π.

Applications of derivatives

① We already calculated equations for tangent lines.

② Derivatives can be used to calculate limits.

Remember $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(h)}{h}$.

So for very small h

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

or

$$h f'(x) \approx f(x+h) - f(x)$$

or

$$f(x) = f(x+h) - h f'(x)$$

Suppose $f(x)$ and $g(x)$ are continuous functions and that for some c we have $f(c) = g(c) = 0$.

Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} \approx \lim_{x \to c} \frac{f(x+h) - h f'(x)}{g(x+h) - h g'(x)}$$

and for x close to c $f(x+h) \approx 0$ and $g(x+h) \approx 0$.

So $\lim_{x \to c} \frac{f(x)}{g(x)} \approx \lim_{x \to c} \frac{f'(x)}{g'(x)}$

L'Hospital' Rule

If $f(x)$ and $g(x)$ are continuous and $f(c)=g(c)=0$, then $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0} \dfrac{f'(x)}{g'(x)}$.

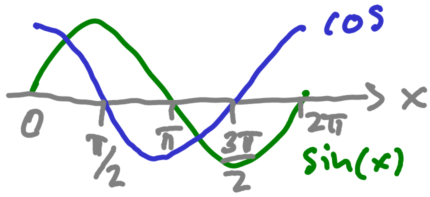
Example: Evaluate $\lim\limits_{x\to \pi/2} \tan(x) - \sec(x)$.

Recall: $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$

$$\sec(x) = \frac{1}{\cos(x)}$$

So $\lim\limits_{x\to\pi/2} \tan(x) - \sec(x) = \lim\limits_{x\to\pi/2} \dfrac{\sin(x)-1}{\cos(x)}$

Recall: $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$



So we use L'Hospital's Rule.

$$\lim_{x\to\pi/2} \frac{\sin(x)-1}{\cos(x)} = \lim_{x\to\pi/2} \frac{\cos(x)}{-\sin(x)} = \frac{0}{-1} = 0$$

Example: Find $\lim\limits_{x\to 0} \dfrac{\sin(x)\cos(x) - x}{x^3}$.

Use L'Hospital $\lim\limits_{x\to 0} \dfrac{\sin(x)\cos(x)-x}{x^3} = \lim\limits_{x\to 0} \dfrac{\cos^2(x) - \sin^2(x) - 1}{3x^2}$

$$= \lim_{x\to 0} \frac{-4\cos(x)\sin(x)}{6x} = \lim_{x\to 0} \frac{4\sin^2(x) - 4\cos^2(x)}{6} = -\frac{2}{3}$$