Maxima, Minima and Saddle Points

So let's find the real shape of the graph of

$$f(x) = \frac{2x^2}{(x-1)(x+2)}$$

A critical point of a function is a zero/root of its derivative.

Calculating derivatives

(1) Some derivatives you should know:

$f(x)$	$f'(x) = \frac{df}{dx}(x)$
x^n	$n x^{n-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$

(2) Rules to calculate more difficult derivatives

Sums: $\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{df}{dx}(x) + \frac{dg}{dx}(x)$

Constant factor: $\frac{d}{dx}\left(K f(x)\right) = K \frac{df}{dx}(x)$, $K = \text{const.}$

Product Rule: $\frac{d}{dx}\left(f(x)\, g(x)\right) = \frac{df}{dx}(x)\, g(x) + f(x)\, \frac{dg}{dx}(x)$

Chain Rule: $\frac{d}{dx}\left(f(g(x))\right) = \frac{df}{dx}(g(x))\,\frac{dg}{dx}(x)$

Quotient Rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{df}{dx}(x)\,g(x) - f(x)\,\frac{dg}{dx}(x)}{[g(x)]^2}$

Examples: * $\frac{d}{dx}\left(\sin^2(x)\right) = ?$

let $f(x) = x^2$ and $g(x) = \sin(x)$,

then $f(g(x)) = \sin^2(x)$ and by the chain rule

$$\frac{d}{dx}\left(\sin^2(x)\right) = 2\sin(x)\cos(x)$$

* $\frac{d}{dx}\left(\ln(3x^2+4x-2)\right)$. With $f(x) = \ln(x)$

and $g(x) = 3x^2+4x-2$, we get that

$f(g(x)) = \ln(3x^2+4x-2)$ and by the chain rule

$$\frac{d}{dx}\left(\ln(3x^2+4x-2)\right) = \frac{1}{3x^2+4x-2}(6x+4)$$

* $\frac{d}{dx}\left(\cos(e^{x^2})\right) = -\sin(e^{x^2})\,e^{x^2}\,2x$

Now back to $f(x) = \frac{2x^2}{(x-1)(x+2)}$

Note: $(x-1)(x+2) = x^2+x-2$

Using the quotient rule $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$,

we get

$$f'(x) = \frac{4x(x^2+x-2) - 2x^2(2x+1)}{(x-1)^2(x+2)^2}.$$

The critical points of f are the zeros of f'

So solve $4x(x^2+x-2) - 2x^2(2x+1) = 0$

$$= \underline{4x^3} + 4x^2 - 8x - \underline{4x^3} - 2x^2$$

$$= 2x^2 - 8x = x(2x-8)$$

So $x=0$ or $2x-8=0$, i.e. $x=4$.