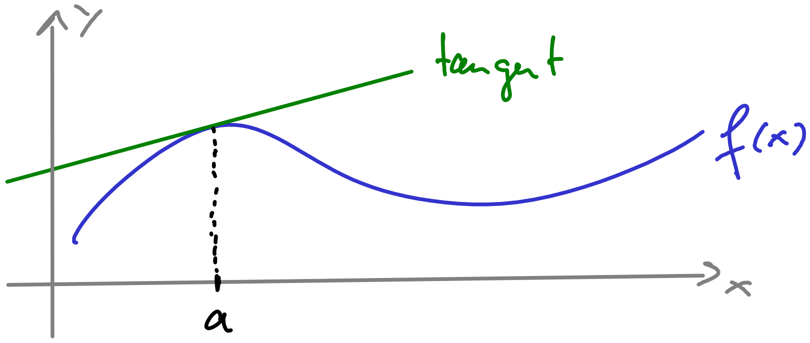
Tangent lines

Problem: Given a continuous function $f(x)$, find the equation for the tangent line to the graph of f at $(a, f(a))$.



Sol^n: The tangent line has slope $f'(a) = \frac{df}{dx}(a)$, so is given by

$$t(x) = f'(a)x + b$$

satisfying $t(a) = f(a)$. So $b = f(a) - f'(a)a$

Roots of functions

Definition: A root of a function $f(x)$ is a number a s.t. $f(a) = 0$.

In other words, an x-intercept of f.

Example: Find the roots of

$$f(x) = 6x^2 + 30x + 18$$

Sol^n: Solve $6x^2 + 30x + 18 = 0$ for x.

Divide by 6: $x^2 + 5x + 3 = 0$

$$x = -\frac{5}{2} \pm \sqrt{\frac{25}{4} - 3} = -\frac{5 \pm \sqrt{13}}{2}$$

Example: How many real roots, does

$x^3 - 2x^2 = 20$ have?

Rmk: A sum of multiples of powers of x is called a polynomial.

The highest power is the degree of the polynomial.

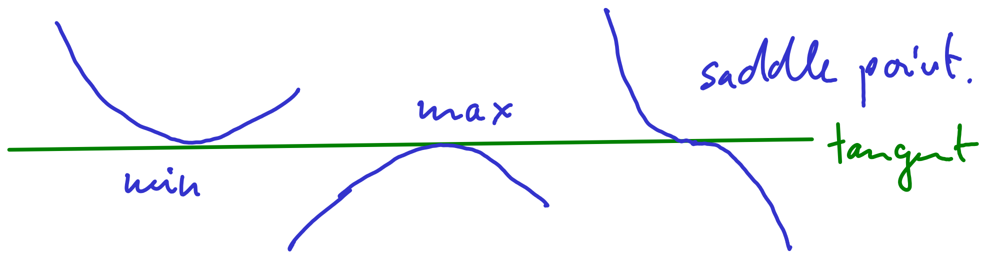
Theorem: A polynomial of degree d has at most d complex roots.

Back to the example: Let write $f(x) = x^3 - 2x^2 - 20$, so we are looking for the roots of f.

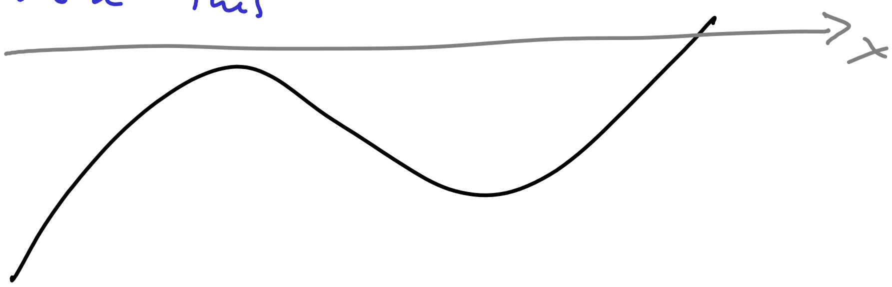
$$\left.\begin{array}{ll} \text{For } x \to \infty & f(x) \to \infty \\ \text{For } x \to -\infty & f(x) \to -\infty \end{array}\right\} \text{ shows that there is at least one zero / root}$$

$$f'(x) = 3x^2 - 4x = (3x-4)x$$

The zeros/roots of $f'(x)$ correspond to local maxima or minima or saddle points, because slope (of tangent line) equal to zero means horizontal tangent



Now we know that the graph of $f(x) = x^3 - 2x^2 - 20$ looks like this



The min and max of f are at 0 and $4/3$ (the zeros of f')

Now $f(0) = -20$, $f(4/3) = \frac{64}{27} - \frac{32}{9} - 20 < 0$

So, the x-axis is above the local max at $x=0$ and so $f(x)$ has precisely one real root.