Example: Find the derivative of $f(x) = \sin(x)$.

Sol^n: $f'(x) = \lim_{h \to 0} \dfrac{\sin(x+h) - \sin(x)}{h}$

$$= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sin(x)(\cos(h) - 1)}{h} + \frac{\cos(x)\sin(h)}{h} \right)$$

($\cos(h) \to 1$, so $\cos(h) - 1 \to 0$; $\sin(h) \to 0$, so $\dfrac{\sin(h)}{h} \to 1$)

$$= \cos(x)$$

Limits at infinity

Consider $f(x) = \dfrac{x}{\sqrt{x^2+1}}$; it's defined everywhere.

Find $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$

Sol^n: Since $\dfrac{x}{\sqrt{x^2+1}} \cdot \dfrac{1/x}{1/x} = \dfrac{x/x}{\frac{\sqrt{x^2+1}}{x}} = \dfrac{1}{\sqrt{\frac{x^2+1}{x^2}}}$ ($x \geq 0$)

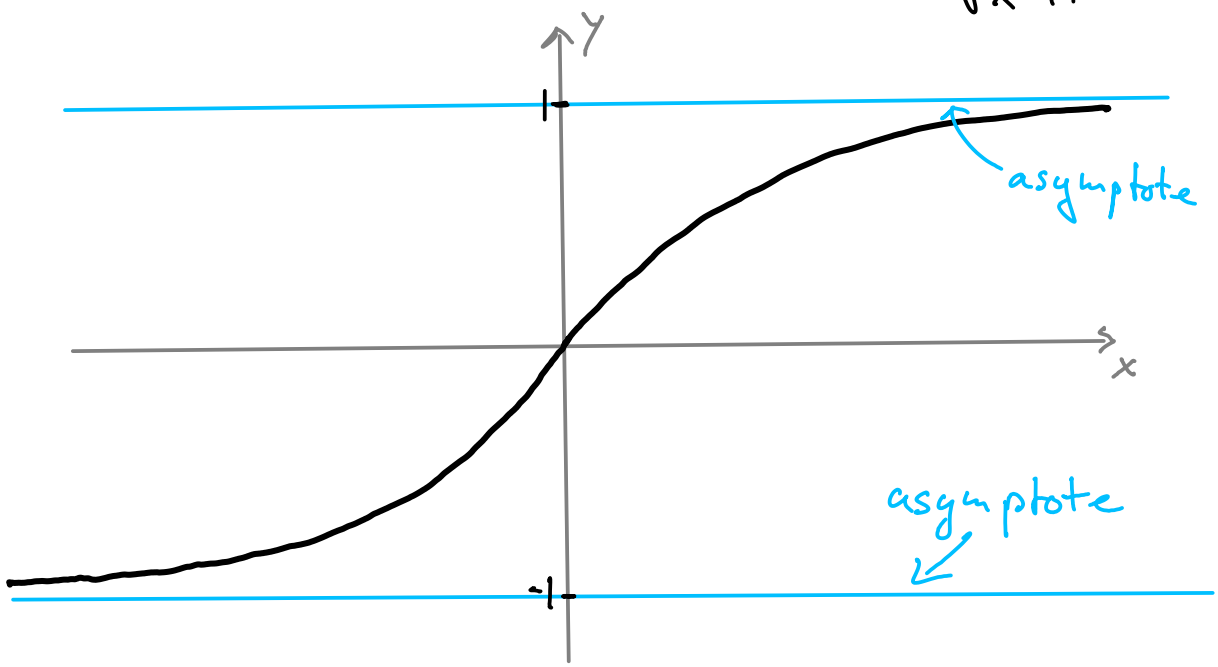
$$= \frac{1}{\sqrt{1 + 1/x^2}} \xrightarrow{x \to \infty} 1$$

So $\lim_{x \to \infty} \dfrac{x}{\sqrt{x^2+1}} = 1$

For large negative x, we get $\dfrac{x}{\sqrt{x^2+1}} = \dfrac{-1}{\sqrt{1+1/x^2}}$.

So $\lim_{x \to -\infty} \dfrac{x}{\sqrt{x^2+1}} = -1$

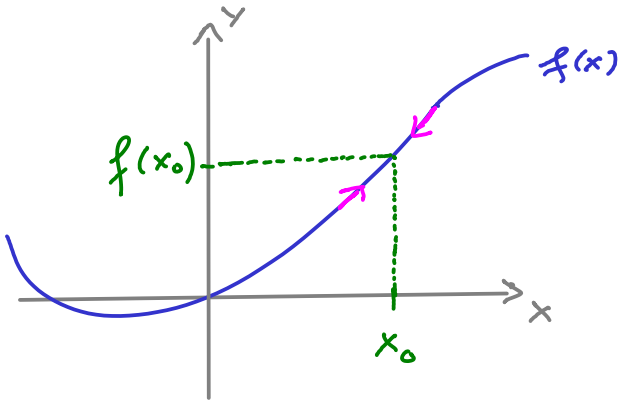
Let's sketch the graph of $f(x) = \dfrac{x}{\sqrt{x^2+1}}$



Continuity

A function $f(x)$ is continuous at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0)$$



Example: What are the asymptotes and discontinuities of $f(x) = \dfrac{2x-5}{3x+2}$?

Sol^n: We find $\lim\limits_{x \to \infty} \dfrac{2x-5}{3x+2} = \dfrac{2}{3}$ because

$$\frac{2x-5}{3x+2} = \frac{2x-5}{3x+2} \cdot \frac{1/x}{1/x} = \frac{2 - 5/x}{3 + 2/x}$$

Similarly $\lim_{x \to -\infty} \frac{2x-5}{3x+2} = \frac{2}{3}$

The discontinuity is where the denominator becomes 0

which is $-\frac{2}{3}$