DIFFERENTIATION

DEFINITION: A function f is a rule which assigns to every real number x a unique real number $f(x)$ or is undefined at x.

DEFINITION: The derivative of a function $f(x)$ is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example: Find the derivative of $f(x) = \sqrt{x}$ $(x \geq 0)$.

Sol^n

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

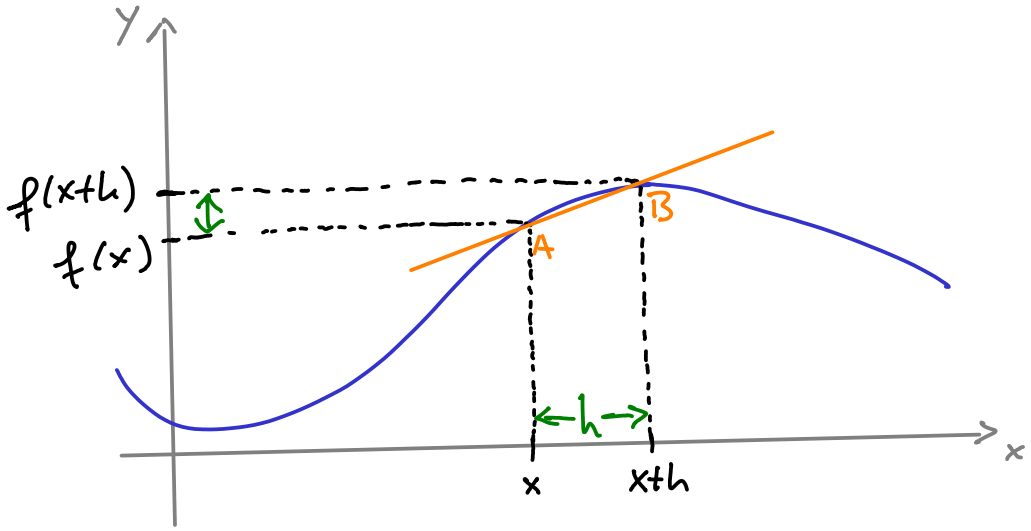
$(a-b)(a+b) = a^2 - b^2$

$$= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Interpretation of derivatives

Consider a function $f(x)$ and its graph.



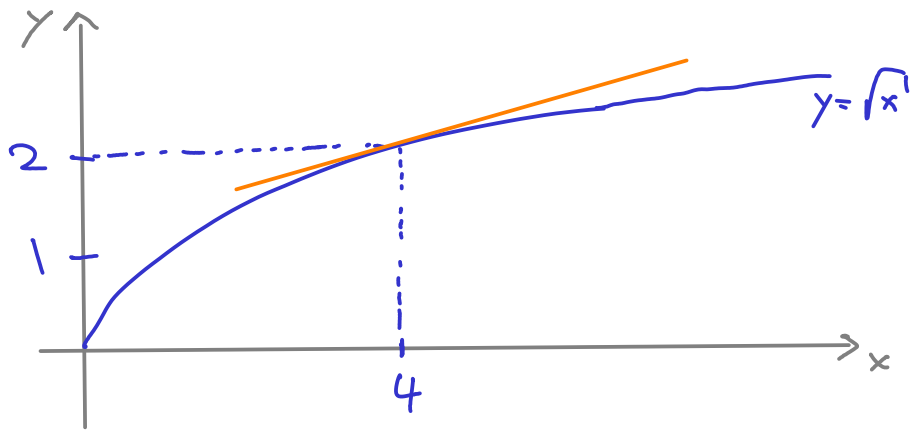
Slope of line AB $= \dfrac{f(x+h)-f(x)}{h}$

In the limit when $h \to 0$ we obtain the slope of the tangent line to the graph of f at the point $(x, f(x)) = A$.

In general: $f'(x)$ is the slope of the graph of f at $(x, f(x))$, if $f'(x)$ exists.

Example: Find the equation of the tangent to the graph of $f(x) = \sqrt{x}$ at $x = 4$.

Sol^n:



At $x = 4$ $f(x) = 2$ and $f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{4}$

In general, a line is given by a function of the form

$$\ell(x) = mx + b$$

(m: slope, b: y-axis intercept)

In the example, the tangent in question is given by $t(x) = \frac{1}{4}x + b$ which needs to contain the point $(4,2)$, i.e. $t(4) = 2$ must hold.

This shows that $b = 1$ and

$$t(x) = \frac{1}{4}x + 1$$