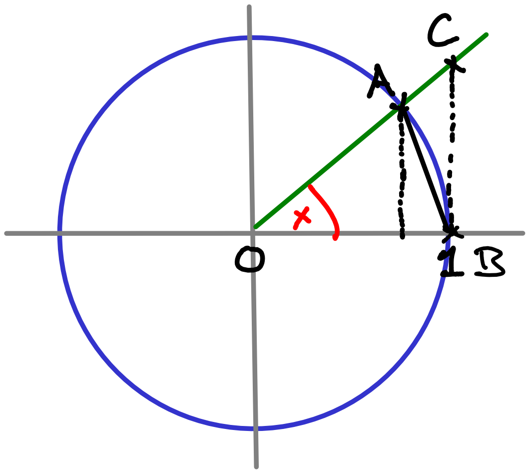
Limits continued



Problem: $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x} = ?$

For small x: Note that

area of triangle OAB $\leq$ area of sector OAB $\leq$ area of triangle OBC

So
$$\frac{1}{2}\sin(x) \leq \frac{1}{2}x \leq \frac{1}{2}\tan(x) = \frac{1}{2}\frac{\sin(x)}{\cos(x)}.$$

So $\quad \sin(x) \leq x \leq \dfrac{\sin(x)}{\cos(x)}$

or $\quad \dfrac{1}{\sin(x)} \geq \dfrac{1}{x} \geq \dfrac{\cos(x)}{\sin(x)}$

Note that for small $x>0$, we have $\sin(x) > 0$.

So $\quad 1 = \dfrac{\sin(x)}{\sin(x)} \geq \dfrac{\sin(x)}{x} \geq \cos(x)$

Now, as $x \to 0$, we have $\cos(x) \to 1$.

Hence
$$\boxed{\lim_{x \to 0} \frac{\sin(x)}{x} = 1}$$

Note: We should consider $x<0$ and small as well.

Example: Evaluate $\lim\limits_{x\to 0} \dfrac{\cos(x)-1}{x}$.

Solution: $\dfrac{\cos(x)-1}{x} = \dfrac{\cos(x)-1}{x} \cdot \dfrac{\cos(x)+1}{\cos(x)+1}$

$$= \frac{\cos^2(x)-1}{x(\cos(x)+1)} = \frac{-\sin^2(x)}{x(\cos(x)+1)} = \frac{-\sin(x)}{x} \cdot \frac{\sin(x)}{\cos(x)+1}$$

So $\lim\limits_{x\to 0} \dfrac{\cos(x)-1}{x} = \lim\limits_{x\to 0} \underbrace{\dfrac{-\sin(x)}{x}}_{\to 1} \cdot \underbrace{\dfrac{\sin(x)}{\cos(x)+1}}_{\to 0} = 0$