Axiomatic Propositional Logic

Axiomatic propositional logic is a formal system consisting of the following three ingredients.

Well formed formulæ (**wff** for short) over the alphabet $\Sigma = \{(,), \neg, \to\} \cup V$, for some arbitrary but fixed countable set V of variables, are defined inductively:

- Every variable $p \in V$ is a **wff**.
- If A and B are **wff**, then so are $(\neg A)$ and $(A \to B)$.
- Nothing else is a **wff**.

Three axiom schemes for any **wff** A, B and C:

$$
\begin{aligned}
&\text{Ax1:} && (A \to (B \to A)) \\
&\text{Ax2:} && ((A \to (B \to C)) \to ((A \to B) \to (A \to C))) \\
&\text{Ax3:} && (((\neg B) \to (\neg A)) \to (A \to B))
\end{aligned}
$$

Deductions are sequences of **wff** in which every term is either an (instance of an) axiom, a hypothesis or obtained from previous terms in the sequence using *Modus Ponens* (MP), namely if A and $(A \to B)$ are in the sequence, then we may append B to it, for any **wff** A and B. That C can be deduced from the hypotheses $A_1, A_2, \dots A_n$ is denoted by $\{A_1, A_2, \dots A_n\} \vdash C$.

	Example dedcutions; $\mathcal{H}$ means hypohtesis	
1.	$A \to ((B \to A) \to A)$	Ax1
2.	$(A \to ((B \to A) \to A)) \to$ $((A \to (B \to A)) \to (A \to A))$	Ax2
3.	$(A \to (B \to A)) \to (A \to A)$	MP 1.&2.
4.	$A \to (B \to A)$	Ax1
5.	$A \to A$ *Note:* now DT and TI hold	MP 4.&3.
6.	$\neg B \to (\neg C \to \neg B)$	Ax1
7.	$(\neg C \to \neg B) \to (B \to C)$	Ax3
8.	$\neg B \to (B \to C)$	TI 6.–7.
9.	$\neg\neg B \to (\neg B \to \neg\neg\neg B)$	Thm 8.
10.	$(\neg B \to \neg\neg\neg B) \to (\neg\neg B \to B)$	Ax3
11.	$\neg\neg B \to (\neg\neg B \to B)$	TI 9.–10.
12.	$\{\neg\neg B\} \vdash B$	2×MP $\mathcal{H}$&11.
13.	$\neg\neg B \to B$	DT 12.
14.	$\neg\neg\neg A \to \neg A$	Thm 13.
15.	$(\neg\neg\neg A \to \neg A) \to (A \to \neg\neg A)$	Ax3
16.	$A \to \neg\neg A$	MP 14.&15

17.	$\{B \to A\} \vdash \neg\neg B \to \neg\neg A$	TI 13.–$\mathcal{H}$–16.
18.	$(\neg\neg B \to \neg\neg A) \to (\neg A \to \neg B)$	Ax3
19.	$\{B \to A\} \vdash \neg A \to \neg B$	MP 17.&18.
20.	$(B \to A) \to (\neg A \to \neg B)$	DT 19.
21.	$\{B,\ B \to C\} \vdash C$	MP on $\mathcal{H}$
22.	$B \to ((B \to C) \to C)$	2×DT 21.
23.	$((B \to C) \to C) \to (\neg C \to \neg(B \to C))$	Thm 20.
24.	$B \to (\neg C \to \neg(B \to C))$	TI 22.–23.
25.	$\neg A \to (A \to \neg X)$	Thm 8.
26.	$(\neg A \to (A \to \neg X)) \to ((\neg A \to A) \to (\neg A \to \neg X))$	Ax2
27.	$(\neg A \to A) \to (\neg A \to \neg X)$	MP 25.&26.
28.	$(\neg A \to \neg X) \to (X \to A)$	Ax3
29.	$(\neg A \to A) \to (X \to A)$	TI 27.–28.
30.	$\{\neg A \to A\} \vdash A$ put $X = \neg A \to A$ and use	2×MP $\mathcal{H}$&29.
31.	$(\neg A \to A) \to A$	DT 30.
32.	$\{B \to A,\ \neg B \to A\} \vdash \neg A \to A$	TI 19.–$\mathcal{H}$
33.	$\{B \to A,\ \neg B \to A\} \vdash A$	MP 31.&32.
34.	$(B \to A) \to ((\neg B \to A) \to A)$	2×DT 33.

Deduction Theorem (DT). If $\Delta \subseteq \mathcal{W}$ and $A \in \mathcal{W}$, then $\Delta \cup \{A\} \vdash B$ if and only if $\Delta \vdash A \to B$.

Sketch proof. "If" follows by adding A as a hypothesis and MP. "Only if" is shown by induction on the length of a deduction sequence for B. If B is a hypothesis or an axiom, then use either $\vdash A \to A$ [5.] (when $B = A$) or $\vdash (B \to (A \to B))$ [Ax1] and MP. Otherwise B is obtained from C and $C \to B$ by MP and the induction hypothesis gives $\Delta \vdash A \to C$ and $\Delta \vdash A \to (C \to B)$. Use Ax2 and twice MP to get $\Delta \vdash A \to B$.

Transitivity of Implication (TI). $\{A \to B,\ B \to C\} \vdash A \to C$ (Follows easily using DT.)

A *(truth) valuation* is a function $v\colon \mathcal{W} \to \{\mathsf{T}, \mathsf{F}\}$, where $\mathcal{W}$ denotes the set of **wff**, satisfying $v(A) \neq v(\neg A)$ and $v(A \to B) = \mathsf{F}$ if and only if $v(B) = \mathsf{F}$ and $v(A) = \mathsf{T}$. A **wff** A is a *tautology* ($\models A$ in symbols), if $v(A) = \mathsf{T}$ for every valuation v.

Theorem. Axiomatic Propositional Logic is *sound* ($\vdash A$ implies $\models A$), *consistent* (not both $\vdash A$ and $\vdash \neg A$) and *complete* ($\models A$ implies $\vdash A$).

Sketch of proof. Soundness is by induction on the length of a deduction sequence for A.

Induction Base: $v(\mathsf{Ax1}) = \mathsf{F} \Longrightarrow v(A) = \mathsf{T}\ \&\ v(B \to A) = \mathsf{F} \Longrightarrow v(A) = \mathsf{T}\ \&\ v(A) = \mathsf{F}$, contradiction

$v(\mathsf{Ax2}) = \mathsf{F} \Longrightarrow v(A \to (B \to C)) = v(A \to B) = v(A) = \mathsf{T}\ \&\ v(C) = \mathsf{F} \Longrightarrow v(B) = v(B \to C) = \mathsf{T} \Longrightarrow v(C) = \mathsf{T}$, contradiction

$v(\mathsf{Ax3}) = \mathsf{F} \Longrightarrow v(A \to B) = \mathsf{F}\ \&\ v(\neg B \to \neg A) = \mathsf{T} \Longrightarrow v(A) = \mathsf{T}\ \&\ v(B) = \mathsf{F}\ \&\ v(\neg B) = \mathsf{F}$, contradiction

Induction Step: $v(A) = \mathsf{T}\ \&\ v(A \to B) = \mathsf{T} \Longrightarrow v(B) = \mathsf{T}$, so if A and $(A \to B)$ are tautologies, then so is B

Consistency now follows, since $\vdash A$ and $\vdash \neg A$ implies $v(A) = v(\neg A) = \mathsf{T}$ for every valuation v, contradicting the definition of a valuation. Completeness is a consequence of Lemmas 1 and 2 below, which allow elimination of all q_i if A is a tautology, by choosing valuations u, v with $u(q_i) = \mathsf{T}$ and $v(q_i) = \mathsf{F}$.

Lemma 1. Let v be valuation and A a **wff** with set of variables $p_1, p_2 \dots, p_n$. Then

$$\{q_1, q_2, \dots, q_n\} \vdash \bar{A}, \text{ where } q_i = \begin{cases} p_i, & \text{if } v(p_i) = \mathsf{T} \\ \neg p_i, & \text{if } v(p_i) = \mathsf{F} \end{cases} \quad \text{and} \quad \bar{A} = \begin{cases} A, & \text{if } v(A) = \mathsf{T} \\ \neg A, & \text{if } v(A) = \mathsf{F} \end{cases}.$$

Proof. Induction on the number of logical operators $\neg$ and $\to$ in A; $\mathcal{IH}$ means induction hypothesis.

Induction base: $A = p_1$
- $v(A) = v(p_1) = \mathsf{T}$, then $\bar{A} = A$ and $q_1 = p_1$, but clearly $\{p_1\} \vdash p_1$
- $v(A) = v(p_1) = \mathsf{F}$, then $\bar{A} = \neg A$ and $q_1 = \neg p_1$, but clearly $\{\neg p_1\} \vdash \neg p_1$

Induction step: $A = \neg B$
- $v(A) = \mathsf{T}$, then $\bar{A} = A$, $\bar{B} = \neg B$ and $\{q_1, \dots, q_n\} \vdash \bar{B}$ by $\mathcal{IH}$, but $\bar{A} = \bar{B}$
- $v(A) = \mathsf{F}$, then $\bar{A} = \neg A$, $\bar{B} = B$ so $\mathcal{IH}$ gives $\{q_1, \dots, q_n\} \vdash B$, but $\vdash B \to \neg\neg B$ [16.]

$A = B \to C$
- $v(A) = \mathsf{T}\ \&\ v(B) = \mathsf{F}$, then $\bar{A} = A$, $\bar{B} = \neg B$ so $\mathcal{IH}$ gives $\{q_1, \dots, q_n\} \vdash \bar{B}$, but $\vdash \neg B \to (B \to C)$ [8.]
- $v(A) = \mathsf{T} = v(B)$, then $v(C) = \mathsf{T}$, $\bar{A} = A$, $\bar{C} = C$ so $\{q_1, \dots, q_n\} \vdash C$ by $\mathcal{IH}$, but $\vdash C \to (B \to C)$ [Ax1]
- $v(A) = \mathsf{F}$, then $\bar{A} = \neg A$, $\bar{C} = \neg C, \bar{B} = B$, so $\{q_1, \dots, q_n\} \vdash \neg C, B$ by $\mathcal{IH}$, but $\vdash B \to (\neg C \to \neg(B \to C))$ [24.]

Lemma 2. If $\{q_1, q_2, \dots, q_n\} \vdash A$ and $\{\neg q_1, q_2, \dots, q_n\} \vdash A$, then $\{q_2, \dots, q_n\} \vdash A$.

Proof. This follows from DT and $\vdash (B \to A) \to ((\neg B \to A) \to A)$ [34.], with $B = q_1$.