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Analysis of Grazing Bifurcations within a Discontinuity-geometry Framework

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Introduction

Inherent discontinuities in a *piecewise-smooth* (PWS) dynamical system can cause considerable difficulties in forming a qualitative picture of how the behavior of such a system changes under parameter variation, with an extra range of behaviors not shown by smooth systems, such as period-adding and robust chaos [1].
One type of PWS system is a *periodically-forced impact oscillator* (PFIO), which has two fundamental components, a smooth dynamical system that models the system in free-flight between impacts and a reset rule that models the impact. It is the discontinuities introduced by the reset rule that make PFIOs non-typical and hard to analyse. This study is exploring a geometric methodology - Discontinuity-Geometry [2] - to assist in both analysing and understanding PFIOs.

We will apply discontinuity-geometry to the PFIO

$$\ddot{x} + 2\mu\dot{x} + kx = h\cos\Big(\omega(t+\tau)\Big), \quad x > c, \tag{1}$$
$$\dot{x}(t_i^+) = -r\dot{x}(t_i^-), \quad x = c, \tag{2}$$

where $x(t), \dot{x}(t)$ and $\ddot{x}(t)$ are the position, velocity and acceleration, respectively at time t, μ is the damping coefficient, k is the spring coefficient, h and ω are the amplitude and frequency respectively of the forcing, τ is the initial phase, c is the location of the impact surface, and $r \in (0,1)$ is the coefficient of restitution for an impact occuring at $t = t_i$. With $x(0) = c$ and $\dot{x}(0) = v \geq 0$, and providing $\mu \neq 0$ or $k \neq \omega^2$, (1) has the free-flight solution

$$x_c(\tau, v; t) = He^{-\mu t} + \frac{h(k-\omega^2)}{(k-\omega^2)^2+(2\mu\omega)^2}\cos\Big(\omega(t+\tau)\Big) + \frac{2h\mu\omega}{(k-\omega^2)^2+(2\mu\omega)^2}\sin\Big(\omega(t+\tau)\Big) \tag{3}$$

where H is dependant on the *initial conditions* (IC) and the relative strengths of the damping and spring coefficients.
Here the fixed parameter values $k = h = 1,\ \omega = 1.8$ and $r = 0.8$ are used. The other two parameters, μ and c, will be varied. We will denote an orbit with m impacts in n forcing periods as a (m, n)-orbit.

The Discontinuity-Geometry Framework

This methodology depends upon the form, properties and intersections of a number of system-specific geometric objects in an $\mathbb{R}^n$ representation space. This system, with one degree of freedom and one forcing will generate objects embedded in $\mathbb{R}^3$, with τ, t and v as its axes.

- The fundamental object in this framework is the **Discontinuity Surface** V_c. This can be thought of as a representation of all those points in the space where a trajectory that is at the location of the impact surface with velocity v and phase τ at time $t = 0$ would return to that location - *in the absence of the impact surface*. More formally,
$$V_c = \{(\tau, t, v) \in \mathbb{R}^3 \mid x_c(\tau, v; t) = c\}. \tag{4}$$
- The **Horizon Surface** Z_0 is defined similarly to V_c, but in terms of points where the actual velocity is zero. Thus
$$Z_0 = \{(\tau, t, v) \in \mathbb{R}^3 \mid \dot{x}_c(\tau, v; t) = 0\}. \tag{5}$$
- The **Horizon Curve** H_c is the intersection of V_c and Z_0, and thus indicates where an impact could occur at zero velocity - i.e. a graze.
- The **Period-n Planes** Π_n are the planes where the time is an integer multiple of the period of the forcing.
- The **Period-n Curves** T_n are the intersections of the Π_n planes with V_c.
- Any object that is a subset of V_c may be used to define a **Reset Object** by applying the reset rule to the actual velocity at impact. These will be denoted by prefixing 'R' to to the symbol for the object. These are the only objects used in this framework so far that are dependant on the details of the reset rule.

Physical Relevance

Only a small part of the entire representation space has any physical relevance, and so any intersection between the geometric objects will only be manifested as the dynamical behaviour it represents if it lies in a physically relevant part of the representation space.
The criteria for physical relevance for this system are;

- $v \geq 0$: The definition of the system requires that all trajectories must start with a non-negative velocity.
- $t \geq 0$: This system can display the phenomenon of chattering, and so there is not always a unique solution in backward time. This limits the application of this framework when $t < 0$.
- $t \leq t_i$: As soon as a trajectory returns to V_c when $t = t_i$ the reset rule is applied, and thus all those parts of the representation space that are beyond V_c (as viewed from Π_0) have no physical interpretation.

A further object, the **Grazing Curve** P_c, may now be defined as the physically relevant part of H_c projected onto Π_0

Two Dynamic Scenarios

Under parameter variation, the system (1,2)can manifest a wide range of behaviours as the position of the impact surface is varied through the grazing bifurcation that destroys the non-impacting orbit that occurs at low values of c. We will use the discontinuity-geometry framework to both analyse and explain some of these different behaviours manifested for two damping scenarios - scenario 1 is underdamped with $\mu = 0.5$ and scenario 2 is critically damped with $\mu = 1$.
Figures (1a) and (2a) show a bifurcation diagram of impact surface location vs. maximum separation from the impact surface. These show the system manifesting quite different behaviour in the two scenarios;

- **Scenario 1:** Shortly below the graze, at $c = c_f$, a stable/unstable pair of $(1,2)$-orbits are born at a saddle-node bifurcation. These coexist with the $(0,1)$-orbit until the unstable orbit that meets the non-impacting orbit at the grazing bifurcation.
- **Scenario 2:** There is no saddle-node bifurcation manifested and it is a stable $(1,3)$ orbit that appears at the grazing bifurcation.

Figures (1b-f) and (2b-f) then show the relevant T_n and its reset object RT_n for varying c. T_n shows where an impact occurs after n forcing periods and any intersection between T_n and RT_n shows where a trajectory will reset back to its initial velocity - thus these intersections (marked with a red circle) show where a $(1, n)$ orbit could potentially be manifested.

Figure 1: $\mu = 0.5$ - underdamped.

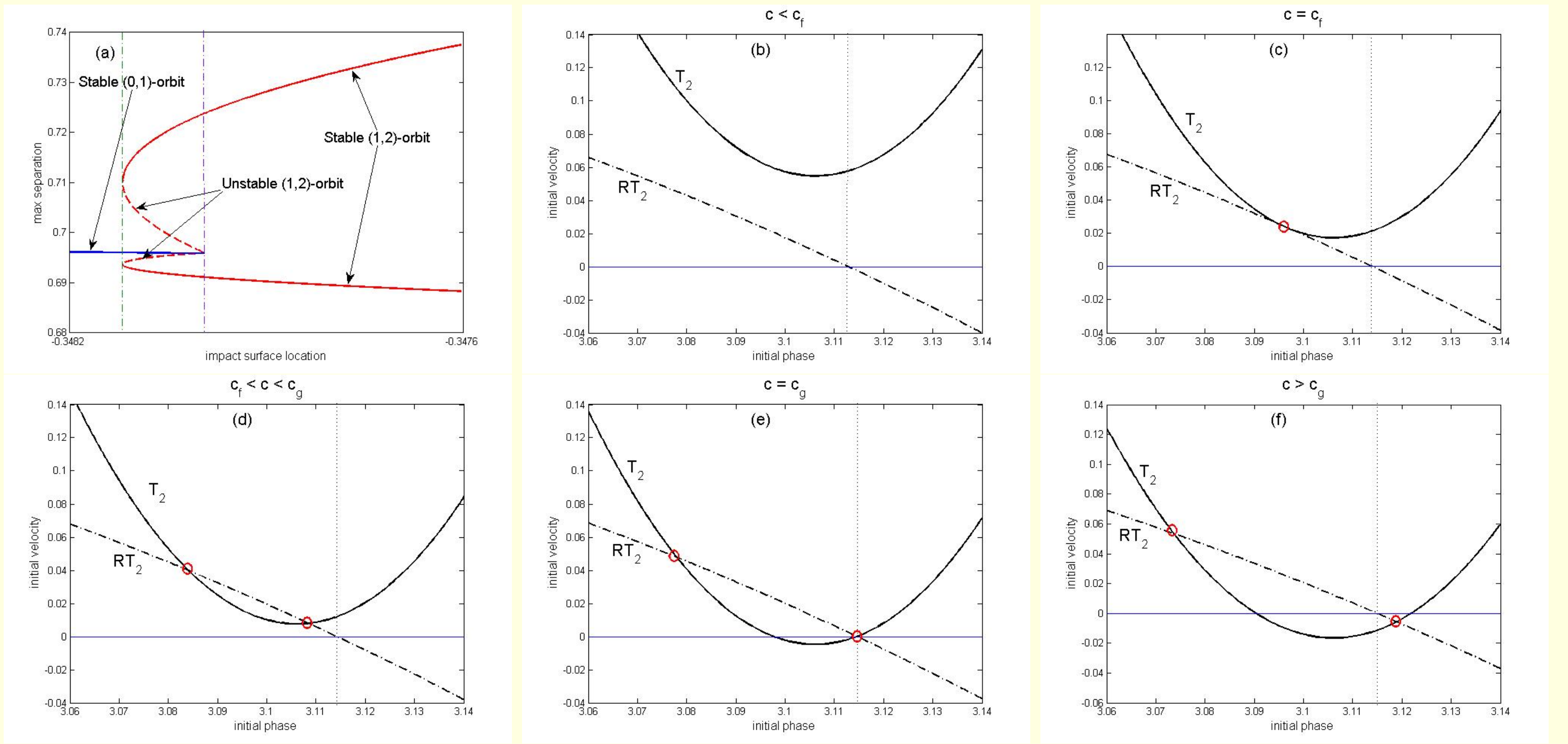
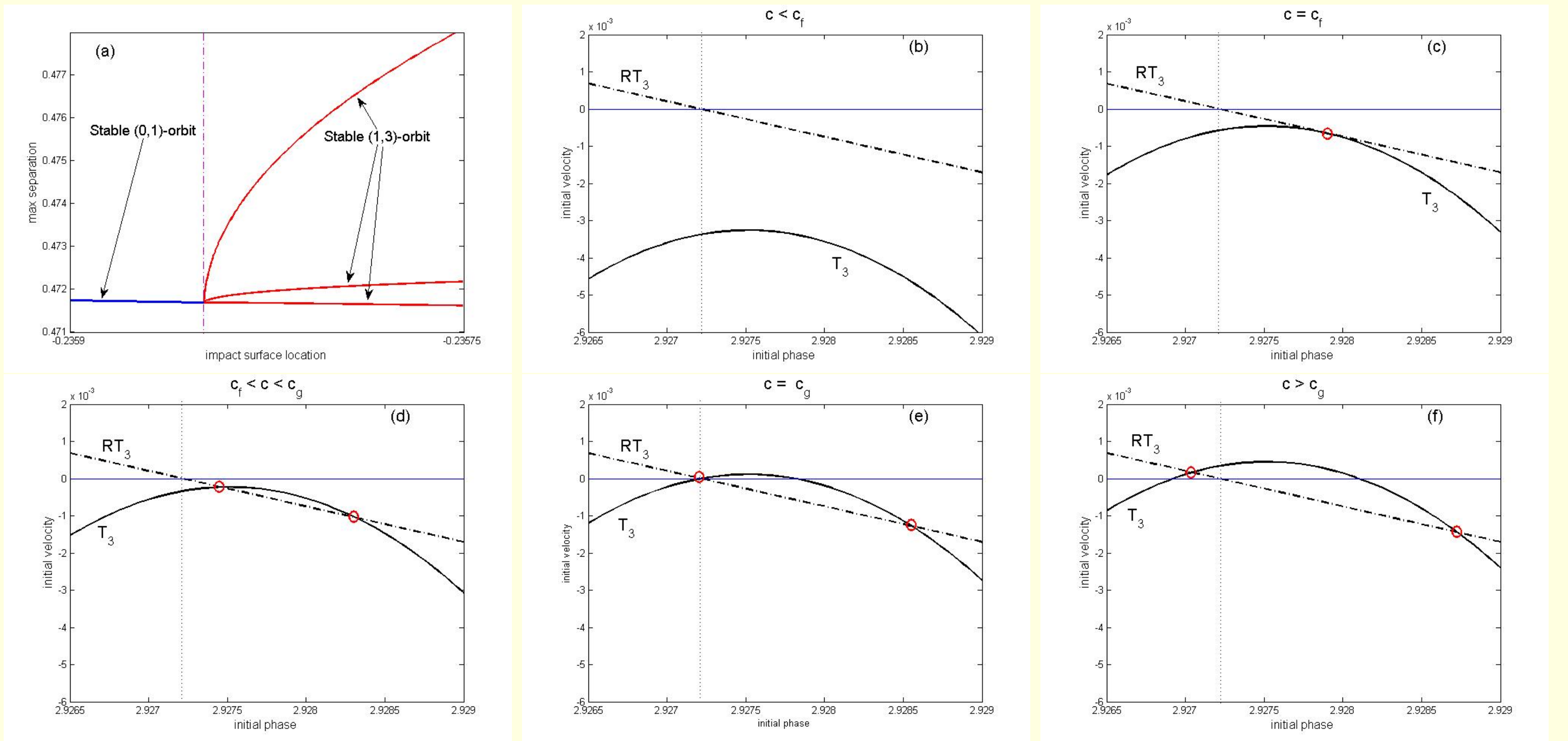


Figure 2: $\mu = 1.0$ - critically damped.



Conclusions

The two sequences of T_n-RT_n diagrams above show exactly the same series of intersections occuring between T_n and RT_n: (b) no intersection, (c) tangential contact at one point of intersection, (d-f) two intersections - in (d) both are the same side of the line $v = 0$, in (e) one is on this line and in (f) there is one on each side of the line.
The differences between the manifested behaviour in the two scenarios is explained by the physical relevance of the intersections. All the intersections that occur when $v < 0$ are not physically relevant, as the statement of this system requires that all trajectories commence with non-negative velocity. Thus in Scenario 1 all the intersections except for one in (f) will be manifested as physical orbits, but in Scenario 2 only one intersection in each of (e) and (f) will be manifested as physical orbits.
This demonstrates that discontinuity-geometry has the capacity to offer explanations fot the differences in behaviour manifested by this system under parameter variation, and not just calculate what these differences will be with a particular set of parameter values.

References

[1] di Bernardo M., Budd C.J., Champneys A.R. and Kowalczyk P., *Piecewise-smooth Dynamical Systems - Theory and Applications*, Springer-Verlag, 2008.

[2] Chillingworth D.R.G., Discontinuity Geometry for an Impact Oscillator, *Dynamical Systems*, pp 389-420, Vol. 4, 2002.