PROFINITE GROUPS WITH RESTRICTED CENTRALIZERS OF COMMUTATORS

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Groups with restricted centralizers are generalizations of FC-groups.

An element $x \in G$ is an FC-element if $|G : C_G(x)|$ is finite, i.e. if $|x^G|$ is finite, where x^G is the set of all conjugates of x in G is finite.

If G is a group, the set $\Delta(G)$ of FC-elements of G is a subgroup, and it is called the FC-center of G.

This happens because $C_G(xy) \geq C_G(x) \cap C_G(y)$ for all $x, y \in G$, so if both $C_G(x)$ and $C_G(y)$ have finite index the same holds for $C_G(xy)$.

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A group G is a FC-group if $G = \Delta(G)$ and it is a BFC-group if it is an FC-group and its conjugacy classes have size bounded by some integer m.

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If G is a profinite FC-group then G' is finite, so G is finite-by-abelian.

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Remark

A profinite finite-by-abelian group is central-by-finite.

This is because if T is a normal finite subgroup of G such that G/T is abelian, then $G' \leq T$. As G is profinite, there exists an open normal subgroup N of G such that $T \cap N = 1$.
Then $[N, G] \leq G' \cap N \leq T \cap N = 1$, so N is contained in the center of G.

Shalev's result is actually more general:

SHALEV,1994

A profinite group with restricetd centralizers is abelian-by-finite. More precisely: the (abstract) subgroup $\Delta(G)$ is closed in G, it has finite index in G and its commutator subroup is finite.

So G is finite-by-abelian-by-finite and thus abelian-by-finite.

A word w on n variables is an element of the free group F with free generators $x_1, \dots, x_n$.
Given a group G, we can think of w as a function $w : G^n \mapsto G$.
We denote by G_w the set of w-values and by $w(G)$ the verbal subgroup generated by G_w.
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Recall that multilinear commutator words, also known as outer commutator words, are words obtained by nesting commutators but using always different variables.
For example the word $[[x_1, x_2], [x_3, x_4, x_5], x_6]$ is a multilinear commutator word but the word $[x_1, x_2, x_2, x_2]$ is not.

Formally, multilinear commutator words are recursively defined as follows:

Definition

The word $w = x_1$ is a multilinear commutator word of weight 1. If u, v are multilinear commutator words of weights m and n respectively involving different variables, then $[u, v]$ is a multilinear commutator word of weight $m + n$.

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Examples

- the lower central words γ_i defined by: $\gamma_1 = x_1$, $\gamma_i = [\gamma_{i-1}, x_i] = [x_1, x_2, \ldots, x_i]$ for $i \geq 1$;
- the derived words δ_i defined by: $\delta_0 = x_1$, $\delta_i = [\delta_{i-1}(x_1, \ldots, x_{2^{i-1}}), \delta_{i-1}(x_{2^{i-1}-1}, \ldots, x_{2^i})]$ for $i \geq 1$.

We obtained a “verbal” version of the result by Shalev.

Theorem (Detomi, M., Shumyatsky)

Let w be a multilinear commutator word and G a profinite group in which all centralizers of w-values are either finite or of finite index. Then $w(G)$ is abelian-by-finite.

The proof of the above result requires some combinatorial techniques for handling multilinear commutators which were introduced by Fernández-Alcober an M. and then developed with Detomi and Shumyatsky.

Proposition

Let $A_1, \ldots, A_n$ be normal subgroups of a profinite group G. Define $\mathcal{X}_w(A_1, \ldots, A_n) = \{w(g_1, \ldots, g_n) | g_i \in A_i \text{ for all } i\}$.
Let H be the topological closure of the abstract subgroup $\Delta(G)$.
If $\mathcal{X}_w(A_1, \ldots, A_n) \subseteq \Delta(G)$ then $[H, w(A_1, \ldots, A_n)]$ is finte.

Proof of the Theorem.

Suppose x is a w-value with infinite order. Then $C_G(x)$ is infinite, hence of finite index.
Let N be an open normal subgroup of G contained in $C_G(x)$, and let $K = K_1 \cdots K_n$, where $K_i = w(G, \ldots, N, \ldots, G)$ (here N appears in the i-th entry).
Let $y = w(g_1 \ldots, u, \ldots, g_n)$, where $u \in N$ appears in the i-th entry.
Then $y \in N \leq C_G(x)$, so x centralizes y and thus $C_G(y)$ is infinite, so $y \in \Delta G$.
It follows from the above proposition that $[H, K_i]$ is finite, thus $[H, K] = \prod_i [H, K_i]$ is finite.
As $K \leq H$ we have that K' is finite.

Moreover, from the fact that N is open in G it follows that K is open in $w(G)$.
So $w(G)$ is finite-by-abelian-by-finite and thus abelian-by-finite.

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Remark

If all w-values are FC-elements we argue as above with $N = G$ and we get that $w(G)'$ is finite.

Indeed, in this case $K = w(G)$.

It can be proved that when all w-values in G have finite order then $w(G)$ is locally finite.

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- It follows from Ore conjecture (Liebeck, O'Brien, Shalev, Tiep, 2010) that in a cartesian product U of finite simple groups every element is a commutator, thus every element is a w-value. Thus we are in a condition where the centralizer of each element is either finite or of finite index and wh apply Shalev's result. So U is abelian-by-finite, thus finite.

- For dealing with the pro-p factors we rely on the techniques developed by Zelmanov for the solution of the restricetd Burnside Problem.

Corollary 1

Let w be a multilinear commutator word and G a profinite group in which all centralizers of w-values are either finite or of finite index. Then G is soluble-by-finite.

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We use

Let w be a multilinear commutator word and G a group. If H is a normal subgroup of G such that $N \cap G_w = 1$ then N centralizes $w(G)$.

Naturally, a corresponding result for finite groups must be of quantitative nature:

Let m be a positive integer, w a group-word, and G a finite group such that $w(G) \neq 1$ and $C_G(x)$ has order at most m for each nontrivial w-value x of G. Does it follows that the order of G is bounded in terms of m and w only?

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It is not difficult to see that for some words w the answer is negative. In particular, this happens when $w = x^n$ is a power word, with $n > 1$.

Example

Let $w = x^3$ and let N be an elememtary abelian 3 group and let a be an involution acting on N by inverting all elements. Then in the semidirect product $G = N \rtimes \langle a \rangle$ all elements ouside N are involutions and they are self-centralizing. The set of nontrivial w-values is precisely $G \setminus N$ and all such elements have a centralizer of order $m = 2$, while the order of G can be arbitrarily large.

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On the other hand, if all nontrivial elements of a finite group G have centralizers of order at most m, then $|G|$ is m-bounded.

Isaacs, 1986

If G is a soluble group where every nontrivial element has a centralizer of order at most m, then G has order at most m^2.

So, the answer is positive for $w = x$.

Detomi, M., Shumyatsky

Let p be a prime, $q_1, \ldots, q_n$ some p-powers and $v = v(x_1, \ldots, x_n)$ a multilinear commutator word of weight at least 2.
Set $w = v(x_1^{q_1}, \ldots, x_n^{q_n})$.
Assume that G is a finite group such that $w(G) \neq 1$ and $|C_G(x)| \leq m$ for every nontrivial w-value x of G.
Then the order of G is (w, m)-bounded.

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The proof uses the following result, due to Hartley, which depends on the classification of finite simple groups.

Hartley 1992

There exists an integer-valued function $f(m)$ such that if G is a finite group containing an element x with $|C_G(x)| \leq m$, then G has a soluble normal subgroup of index at most $f(m)$.

In the case where w is a multilinear commutator word, the result follows easily from Hartley's theorem.
Indeed let T be soluble characteristic subgroup of G of bounded index. Whenever A is a characteristic subgroup of T such that and let i be the smallest integer such that $T^{(i)} \cap G_w = 1$. Then $T^{(i)}$ centralizes $w(G)$ and so $T^{(i)}$ has order at most $|C_G(w(G))| \leq m$. Pass to the quotient over $T^{(i)}$.
Now $A = T^{(i-1)} \lhd G$ is abelian and there exists $x \in A \cap G_w \neq 1$, so both A and $C_G(A)$ have order at most $|C_G(x)| \leq m$ and so $|G| = |C_G(A)||G/C_G(A)| \leq mm!$.

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For $w = v(x_1^{q_1}, \ldots, x_n^{q_n})$, where q_i are p-powers, we use a result by Khukhro: if P is a finite p-group admitting a p-automorphism of order p^s with p^m fixed points, then P has a characteristic (p, s, m)-bounded-index soluble subgroup of (p, s)-bounded derived length.

Recall that the nth Engel word is defined inductively by $[x,_1 y] = [x, y]$, and $[x,_n y] = [[x,_{n-1} y], y]$.

DETOMI, M., SHUMYATSKY

Let w be the nth Engel word or the word $w = [x^k,_n y]$, with $n, k \geq 1$. Assume that G is a finite group such that $w(G) \neq 1$ and $|C_G(x)| \leq m$ for every nontrivial w-value x of G.
Then the order of G is (w, m)-bounded.

Here we use previous results on how the exponents of Sylow p-subgroups of $w(G)$ can bound the exponent of $w(G)$.