Survey of New Developments in Subgroup Growth

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Pyber (2004): Almost every (reasonable) subgroup growth type can be achieved by a 4-generated group.

3 PSG Pro-p Groups

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(vii) $L(G)$ the associated graded Lie algebra is nilpotent.

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(vi) Type $2^{n^{\frac{d-1}{d}}}$, where d is an integer, metabelian pro-p groups (**Klopsch (unpublished)**).

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(a) Are there any other gaps in the subgroup growth types of pro-p groups?

(b) What other subgroup growth types occur for pro-p groups?

(c) Is there an uncountable number of subgroup growth types (up to the necessary equivalence) for pro-p groups?

6 Branch Groups and Types of Subgroup Growth

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7 Just Infinite Pro-p Groups and Types of Subgroup Growth

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Ershov & Jaikin: There are hereditarily just infinite pro-p groups with subgroup growth larger than $n^{(\log n)^{2-\epsilon}}$.

Theorem 2 (B. & Schlage-Puchta): Let $f : \mathbb{N} \to \mathbb{N}$ be a function, such that $f(n) \geq n^3$ and $\frac{\log f(n)}{n} \to 0$. Then there exists a pro-p group G such that $s_n(G)$ is of type $e^{f(\log n)}$.

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Notice: We can obtain any subgroup growth type between $n^{(\log n)^2}=e^{(\log n)^3}$ and $e^n=e^{e^{\log n}}$.

Problem: What types of subgroup growth of pro-p groups exist between $n^{\log n}$ and $n^{(\log n)^2}$?

9 Minimal Growth for Non-p-adic Analytic Pro-p Groups

Mann: What is the smallest k such that if G is a pro-p and $s_n(G) \leq n^{c \log n}$ for almost all n, with $c < k$, then G has PSG.

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Theorem 3 (B., Klopsch, & Schlage-Puchta): For p big enough the Nottingham group has subgroup growth as most $n^{\frac{1}{8} \log n}$. Thus, $k = \frac{1}{8}$. (**Work in Progress.**)

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Previous Work (proved a bit less than what they claimed):

Müller & Schlage-Puchta (2005)

Gerdau (2010 unpublished).

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Corollary C: There exists a f.g. group with characteristic subgroup growth $n^{\log n}$.

Problem: What is the characteristic subgroup growth of a f.g. free (pro-p) group?

12 Slow Normal and Characteristic Subgroup Growths

Theorem 5 (B. & Schlage-Puchta, work in progress): Let $\mu, \eta : \mathbb{N} \to \mathbb{N}$ be non-decreasing functions, such that $\eta(n) \leq \mu(n) < \frac{n}{8}$ for all n and $\eta(n) \to \infty$. Then there exists a f.g. profinite group G such that $s_n^{\triangleleft}(G) \leq \mu(n)^{1+o(1)}$, $s_n^{ch}(G) \leq \eta(n)^{1+o(1)}$ for all n, and there are infinitely many n such that $s_n^{\triangleleft}(G) > \mu(n)$ and infinitely many n such that $s_n^{ch}(G) > \eta(n)$.

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Problem: Find similar results for faster growth.

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Problem: Let G be a pro-p with CNSG. Is it true that the subgroup growth type of G is PSG or $n^{\log n}$?

14 Problems

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(There are only countably many.)

J. S. Wilson (91), Zelmanov (2000), B. & Larsen (99): A finitely presented pro-p group that does not contain a non-abelian free pro-p group has subgroup growth type at most $e^{\sqrt{n}}$. In particular, a finitely presented pro-p group linear over a local field has subgroup growth type at most $e^{\sqrt{n}}$.

15 Problems

Problem: Let P be a finite set of primes. Let $\mathcal{C}$ be

$$\{G \text{ profinite } \mid \text{if } p \mid |G/N|, \text{ where } (G:N) < \infty, \text{ then } p \in P\}.$$

Is there a gap in the spectrum of the subgroup growth type of groups in $\mathcal{C}$?

16 Main Idea of the Proof of Theorem 1

Definitions: Let G be a group we write $d_p(G) = \dim_{\mathbb{F}_p} G/([G,G]G^p)$ and $d_{p,G}(m) = d_p(m) = \max\{d_p(U) \mid (G:U) = p^m\}$.

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Proposition: Let G be a p-group or a pro-p group. If $\mu \leq d_p(m-\mu)$, then

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Proposition: Let G be a p-group, acting transitively on a set X. Let H be the wreath product $G \wr \mathbb{F}_p$ induced by this action. Then we have

$$o_{p^n}(G, X) \leq d_{p,H}(n) \leq d_H(n) \leq o_{p^n}(G, X) + n \max_{m \leq n} d_G(m).$$

Theorem 6 (B. & Schlage-Puchta): Let G be the Grigorchuk group or a Gupta-Sidki group. Let $f : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function, and assume that $\frac{f(n)}{n} \to \infty$, $\frac{\log f(n)}{n} \to 0$. Then there exists a transitive action of G on a set X, such that $o_{p^n}(G, X) \leq f(n)$ for all sufficiently large n, and $o_{p^n}(G, X) \geq \frac{1}{p} f(n)$ for infinitely many n.

Theorem 7 (B. & Schlage-Puchta): Let G be the Grigorchuk group or a Gupta-Sidki group acting on the p-adic tree T. Let Ω be the orbit under G of some infinite path in T. Then there is some C such that $o_{p^m}(G, \Omega) \leq Cm$.

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Theorem 7 (B. & Schlage-Puchta): Let G be the Grigorchuk group or a Gupta-Sidki group acting on the p-adic tree T. Let Ω be the orbit under G of some infinite path in T. Then there is some C such that $o_{p^m}(G, \Omega) \leq Cm$.

Theorem 8 (B. & Schlage-Puchta): Let G be a p-group acting transitively on a set Ω. Suppose $o_n(G, \Omega)$ is unbounded. Then there exist infinitely many m, such that

$$o_{p^m}(G, \Omega) \geq (p-1)m + 1.$$