INTRODUCTION TO FUSION SYSTEMS

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Abstract. This is an expanded lecture note for a series of four talks given by the author for postgraduate students in University of Aberdeen, UK, in February 2010. Logical prerequisites for the main body of the lecture are kept to minimum, namely basic group theory (including Sylow's theorem) and rudiments of category theory (including definitions of category and functor). Full proofs are given whenever possible. But at the same time we present many interesting results and examples without proofs, hoping to indicate the rich connection of the theory of fusion systems with group theory, modular representation theory and homotopy theory.

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1. Definitions and examples

1.1. **Notations and motivations.** First we fix some notations. Throughout this course, p is a prime number. Let G be a finite group. Denote by $\mathrm{Syl}_p(G)$ the set of Sylow p-subgroups of G. For $x \in G$, let $c_x\colon G \to G$ be the conjugation map given by $c_x(u) = xux^{-1}$ for $u \in G$. For $H \leq G$, let ${}^xH = c_x(H) = xHx^{-1}$. For $Q, R \leq G$, let $\mathrm{Hom}_H(Q,R) = \{\varphi\colon Q \to R \mid \varphi = c_x|_Q \text{ for some } x \in H\}$, and $\mathrm{Aut}_H(Q) = \mathrm{Hom}_H(Q,Q)$. Note that $\mathrm{Aut}_H(Q)$ is indeed a subgroup of the full automorphism group $\mathrm{Aut}(Q)$.

Now we introduce our main example.

Definition 1.1. Let P be a Sylow p-subgroup of a finite group G. Let $\mathcal{F}_P(G)$ denote the category whose object set is the set of all subgroups of P and such that for all $Q, R \leq P$,

$$\mathrm{Hom}_{\mathcal{F}_P(G)}(Q,R) = \mathrm{Hom}_G(Q,R),$$

with composition of morphisms given by usual composition of maps.

This category $\mathcal{F}_P(G)$ describes how subgroups of P are related by G-conjugations. Since all Sylow p-subgroups of G are G-conjugate, the category $\mathcal{F}_P(G)$ is equivalent to $\mathcal{F}_{P'}(G)$ for any other Sylow p-subgroup P' of G. So it determines how *all* p-subgroups of G are related by G-conjugations, which is traditionally called the *p-fusion pattern* of G in finite group theory.

The category $\mathcal{F}_P(G)$ has only a small portion of data contained in the group G, but it determines quite a large part of the 'p-local behavior' of the group G. We give two illustrative examples here.

Theorem 1.2 (Frobenius' normal p-complement theorem)**.** *Let G be a finite group with $P \in \mathrm{Syl}_p(G)$. The following are equivalent.*

1. *G is has a normal p-complement, i.e. G has a normal subgroup K such that $G = KP$ and $K \cap P = 1$.*
2. *$N_G(Q)$ has a normal p-complement for every $1 \neq Q \leq P$.*
3. *$\mathrm{Aut}_G(Q)$ is a p-group for every $Q \leq P$.*
4. *$\mathcal{F}_P(G) = \mathcal{F}_P(P)$.*

Sketch of proof. A full proof can be found in [18, 1.4]. (1) ⇒ (2) ⇒ (3) are easy; (3) ⇔ (4) can be shown by Alperin's fusion theorem (Theorem 2.7); (3) ⇒ (1) can be shown by induction on $|G|$ using Alperin's fusion theorem and focal subgroup theorem. □

Theorem 1.3 (Cartan and Eilenberg; [12, XII.10.1])**.** *Let G be a finite group with $P \in \mathrm{Syl}_p(G)$. If M is a $\mathbb{Z}_{(p)}G$-module, then*

$$\begin{aligned} H^*(G,M) &\cong H^*(P,M)^G := \{\alpha \in H^*(P,M) \mid \mathrm{res}^P_{P\cap {}^xP}(\alpha) = \mathrm{res}^{{}^xP}_{P\cap {}^xP}({}^x\alpha)\ \forall x \in G\} \\ &\cong \varprojlim_{\mathcal{F}_P(G)} H^*(-,M). \end{aligned}$$

Sketch of proof. For a full proof, see also [6, 3.8.2]. Elements of $H^*(P,M)^G$ are called the *G-stable elements* of $H^*(P,M)$. The first isomorphism is given by the restriction map $\mathrm{res}^G_P\colon H^*(G,M) \to H^*(P,M)$; to prove that it is an isomorphism, we use the transfer map $\mathrm{tr}^G_P\colon H^*(P,M) \to H^*(G,M)$ and Mackey decomposition formula. The second isomorphism follows from the definition of the inverse limit. □

1.2. **Axioms of saturated fusion systems.** L. Puig axiomatized the category $\mathcal{F}_P(G)$ around 1990. The idea is to forget about the whole group G while keeping the p-group P in sight, and impose certain properties on morphisms between subgroups of P so that they behave as if they are conjugation maps inside some finite group having P as a Sylow p-subgroup. For Puig's exposition of the theory of fusion systems (which he calls *Frobenius categories*), see [21]. Later C. Broto, R. Levi and B. Oliver gave a slightly different, yet equivalent, formulation of the axioms of fusion systems (see [10]), and we are going to follow their language. This requires several steps.

Definition 1.4. Let P be a finite p-group. A *fusion system* on P is a category $\mathcal{F}$ whose object set is the set of all subgroups of P and whose morphism sets consist of injective group homomorphisms (with composition of morphisms give by usual composition of maps) satisfying the following conditions.

(1) $\mathrm{Hom}_{\mathcal{F}}(Q,R) \supseteq \mathrm{Hom}_P(Q,R)$ for all $Q, R \leq P$.
(2) For all $Q, R \leq P$ and all $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$, the induced group isomorphism

$$\overline{\varphi}\colon Q \to \varphi(Q), u \mapsto \varphi(u)$$

and its inverse

$$\overline{\varphi}^{-1}\colon \varphi(Q) \to Q, \varphi(u) \mapsto u$$

are morphisms of $\mathcal{F}$.

When G is a finite group with $P \in \mathrm{Syl}_p(G)$, the category $\mathcal{F}_P(G)$ is clearly a fusion system on P. Let $\mathcal{F}$ be a fusion system on a finite p-group P. As a consequence of its axioms, $\mathcal{F}$ has the following properties:

- $\mathcal{F}$ is a subcategory of the category of all subgroups of P and all injective group homomorphisms between subgroups of P.
- $\mathcal{F}$ contains $\mathcal{F}_P(P)$ as a subcategory.
- If $Q \leq R \leq P$, then the inclusion $Q \hookrightarrow R$ belongs to $\mathcal{F}$.
- Every morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ with $|Q| = |R|$ is an isomorphism in $\mathcal{F}$. In particular, $\mathrm{Aut}_{\mathcal{F}}(Q) := \mathrm{Hom}_{\mathcal{F}}(Q,Q)$ is a subgroup of $\mathrm{Aut}(Q)$.
- Every morphism in $\mathcal{F}$ is the composition of an isomorphism in $\mathcal{F}$ followed by an inclusion.
- $\mathcal{F}$ is closed under restriction both in the domain and in the codomain: if $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ and $Q' \leq Q$, $R' \leq R$ such that $\varphi(Q') \leq R'$, then the induced map $\varphi|_{Q',R'}\colon Q' \to R', u \mapsto \varphi(u)$, belongs to $\mathcal{F}$.

If $Q \leq P$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$, we say that Q and $\varphi(Q)$ are $\mathcal{F}$-*conjugate* and write $Q \cong_{\mathcal{F}} \varphi(Q)$.

Definition 1.5. Let $\mathcal{F}$ be a fusion system on a finite p-group P and let $Q \leq P$.

(1) Q is *fully $\mathcal{F}$-normalized* if $|N_P(Q)| \geq |N_P(Q')|$ for all $Q' \cong_{\mathcal{F}} Q$.
(2) Q is *fully $\mathcal{F}$-centralized* if $|C_P(Q)| \geq |C_P(Q')|$ for all $Q' \cong_{\mathcal{F}} Q$.

Note that every $\mathcal{F}$-conjugacy class of subgroups of P has fully $\mathcal{F}$-normalized subgroups and fully $\mathcal{F}$-centralized subgroups.

Definition 1.6. Let $\mathcal{F}$ be a fusion system on a finite p-group P. For $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$, let

$$N_\varphi := \{y \in N_P(Q) \mid \varphi \circ c_y|_Q \circ \varphi^{-1} \in \mathrm{Aut}_P(\varphi(Q))\}.$$

This is a crucial definition and has to be understood properly. It has to do with the behavior of φ in terms of extensions.

- N_φ is the largest subgroup R of $N_P(Q)$ such that $\varphi \circ \mathrm{Aut}_R(Q) \circ \varphi^{-1} \le \mathrm{Aut}_P(\varphi(Q))$.
- $QC_P(Q) \le N_\varphi \le N_P(Q)$: if $y \in Q$, then for $u \in Q$, $(\varphi \circ c_y \circ \varphi^{-1})(\varphi(u)) = \varphi(yuy^{-1}) = \varphi(y)\varphi(u)\varphi(y)^{-1} = c_{\varphi(y)}(\varphi(u))$; if $y \in C_P(Q)$, then $c_y|_Q = \mathrm{id}_Q$, so $\varphi \circ c_y|_Q \circ \varphi^{-1} = \mathrm{id}_{\varphi(Q)}$.
- Suppose $Q \le R \le N_P(Q)$ and φ *extends* to R (in $\mathcal{F}$), i.e. there exists $\psi \in \mathrm{Hom}_{\mathcal{F}}(R, P)$ such that $\psi|_Q = \varphi$. Then $R \le N_\varphi$ because $\varphi \circ c_y|_Q \circ \varphi^{-1} = c_{\psi(y)}|_{\varphi(Q)}$ for $y \in R$: if $u \in Q$, then $(\varphi \circ c_y \circ \varphi^{-1})(\varphi(u)) = \varphi(yuy^{-1}) = \psi(yuy^{-1}) = \psi(y)\psi(u)\psi(y)^{-1} = c_{\psi(y)}(\varphi(u))$. This shows that N_φ is the largest subgroup of $N_P(Q)$ to which φ can possibly extend.
- More generally, suppose that φ extends to U with $Q < U \le P$. Then $Q < N_U(Q) \le N_P(Q)$ and φ extends to $N_U(Q)$. Thus $N_U(Q) \le N_\varphi$. This means that if φ extends to a subgroup of P properly containing Q, then $N_\varphi > Q$. But the converse does not hold necessarily.

Definition 1.7. Let $\mathcal{F}$ be a fusion system on a finite p-group P. We say that $\mathcal{F}$ is *saturated* if it satisfies the following conditions.

(1) (Sylow axiom) If $Q \le P$ is fully $\mathcal{F}$-normalized, then Q is fully $\mathcal{F}$-centralized and $\mathrm{Aut}_P(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$.
(2) (Extension axiom) For every $Q \le P$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ such that $\varphi(Q)$ is fully $\mathcal{F}$-centralized, φ extends to N_φ.

Note that if $\mathcal{F}$ is a saturated fusion system on a finite p-group P and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ with $\varphi(Q)$ fully $\mathcal{F}$-centralized, then N_φ is the largest subgroup of $N_P(Q)$ to which φ extends, and that φ extends to a subgroup of P properly containing Q if and only if $N_\varphi > Q$.

There are several simplifications of the saturation axioms, which are sometimes very useful. See in particular R. Stancu's version in [18, 2.4,2.5].

1.3. **Fusion systems of finite groups.** We show that the fusion system $\mathcal{F}_P(G)$ of a finite group G with $P \in \mathrm{Syl}_p(G)$ is saturated. First we need to understand what fully normalized and fully centralized mean in this setting.

Proposition 1.8. *Let G be a finite group with $P \in \mathrm{Syl}_p(G)$ and $Q \le P$.*

(1) *Q is full $\mathcal{F}_P(G)$-normalized iff $N_P(Q) \in \mathrm{Syl}_p(N_G(Q))$.*
(2) *Q is full $\mathcal{F}_P(G)$-centralized iff $C_P(Q) \in \mathrm{Syl}_p(C_G(Q))$.*

Proof. (1) Suppose $N_P(Q) \in \mathrm{Syl}_p(N_G(Q))$. Let $\varphi = c_x|_Q \in \mathrm{Hom}_{\mathcal{F}_P(G)}(Q, P)$, $x \in G$. Then $N_P(\varphi(Q)) = N_P({}^xQ) = {}^xN_{{}^{x^{-1}}P}(Q)$, so $|N_P(\varphi(Q))| = |N_{{}^{x^{-1}}P}(Q)|$. Since $N_{{}^{x^{-1}}P}(Q)$ is a p-subgroup of $N_G(Q)$, it follows that $|N_P(Q)| \ge |N_P(\varphi(Q))|$. Thus Q is fully $\mathcal{F}$-normalized.

Conversely, suppose that Q is fully $\mathcal{F}$-normalized. Since $N_P(Q)$ is a p-subgroup of $N_G(Q)$, it is contained in a Sylow p-subgroup S of $N_G(Q)$. Since S is a p-subgroup of G and $P \in \mathrm{Syl}_p(G)$, there is $x \in G$ such that ${}^xS \le P$. Then $Q \le N_P(Q) \le S$, so ${}^xQ \le {}^xS \le P$. Hence $c_x \colon G \to G$ restricts to $\varphi := c_x|_{Q,P} \in \mathrm{Hom}_{\mathcal{F}_P(G)}(Q, P)$. By Sylow's theorem, ${}^xS \le P$ for some $x \in G$. Then ${}^xQ \le {}^xS \le P$, so $c_x|_Q \in \mathrm{Hom}_{\mathcal{F}_P(G)}(Q, P)$. On the other hand, $N_P(Q) \le S \le N_G(Q)$ implies that ${}^xN_P(Q) \le {}^xS \le {}^xN_G(Q) = N_G({}^xQ)$, and so ${}^xN_P(Q) \le {}^xS \le N_P({}^xQ)$. But $|{}^xN_P(Q)| = |N_P(Q)| \ge |N_P({}^xQ)|$ by assumption. Thus $N_P(Q) = S \in \mathrm{Syl}_p(N_G(Q))$.

(2) Suppose $C_P(Q) \in \mathrm{Syl}_p(C_G(Q))$. Let $\varphi = c_x|_Q \in \mathrm{Hom}_{\mathcal{F}_P(G)}(Q, P)$, $x \in G$. Then $C_P(\varphi(Q)) = C_P({}^xQ) = {}^xC_{x^{-1}P}(Q)$, so $|C_P(\varphi(Q))| = |C_{x^{-1}P}(Q)|$. Since $C_{x^{-1}P}(Q)$ is a p-subgroup of $C_G(Q)$, it follows that $|C_P(Q)| \geq |C_P(\varphi(Q))|$. Thus Q is fully $\mathcal{F}$-centralized.

Conversely, suppose that Q is fully $\mathcal{F}$-centralized. Since $C_P(Q)$ is a p-subgroup of $C_G(Q)$, it is contained in a Sylow p-subgroup S of $C_G(Q)$. Since S normalizes Q, SQ is a subgroup of G, and in particular a p-subgroup of G. By Sylow's theorem, there is $x \in G$ such that ${}^x(SQ) \leq P$. Then ${}^xQ \leq P$, so $c_x \colon G \to G$ restricts to $\varphi := c_x|_{Q,P} \in \mathrm{Hom}_{\mathcal{F}_P(G)}(Q, P)$. On the other hand, $C_P(Q) \leq S \leq C_G(Q)$ implies that ${}^xC_P(Q) \leq {}^xS \leq {}^xC_G(Q) = C_G({}^xQ)$, and so ${}^xC_P(Q) \leq {}^xS \leq C_P({}^xQ)$. But $|{}^xC_P(Q)| = |C_P(Q)| \geq |C_P({}^xQ)|$ by assumption. Thus $C_P(Q) = S \in \mathrm{Syl}_p(C_G(Q))$. ☐

Proposition 1.9. *Let G be a finite group with $P \in \mathrm{Syl}_p(G)$. Then $\mathcal{F}_P(G)$ is a saturated fusion system on P.*

Proof. We first prove the Sylow axiom. Let $Q \leq P$ be fully $\mathcal{F}$-normalized. We have $\mathrm{Aut}_G(Q) \cong N_G(Q)/C_G(Q)$ and $\mathrm{Aut}_P(Q) \cong N_P(Q)/C_P(Q)$. Since $N_P(Q) \in \mathrm{Syl}_p(N_G(Q))$ by (1), we have that

$$\frac{|N_G(Q)|}{|N_P(Q)|} = \frac{|C_G(Q)|}{|C_P(Q)|} \cdot \frac{|\mathrm{Aut}_G(Q)|}{|\mathrm{Aut}_P(Q)|}$$

is prime to p. It follows that $C_P(Q) \in \mathrm{Syl}_p(C_G(Q))$ and $\mathrm{Aut}_P(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_G(Q))$. By (2), the former means that Q is fully $\mathcal{F}_P(G)$-centralized.

To prove the extension axiom, let $\varphi \in \mathrm{Hom}_{\mathcal{F}_P(G)}(Q, P)$ with $R := \varphi(Q)$ fully $\mathcal{F}_P(G)$-centralized. Then there exists $x \in G$ such that $\varphi = c_x|_Q$ and $C_P(R) \in \mathrm{Syl}_p(C_G(R))$ by (1). For $y \in N_P(Q)$, we have $y \in N_\varphi$ iff $\varphi \circ c_y \circ \varphi^{-1} = c_{xyx^{-1}}|_R \in \mathrm{Aut}_P(R)$ iff $xyx^{-1} \in N_P(R)C_G(R)$. Thus ${}^xN_\varphi \leq N_P(R)C_G(R)$. Now $N_P(R) \in \mathrm{Syl}_p(N_P(R)C_G(R))$ and ${}^xN_\varphi$ is a p-subgroup of $N_P(R)C_G(R)$. By Sylow's theorem, there are $n \in N_P(R)$ and $c \in C_G(R)$ such that ${}^{ncx}N_\varphi \leq N_P(R)$, i.e. ${}^{cx}N_\varphi \leq N_P(R)$. Let $\psi = c_{cx} \colon N_\varphi \to P$. Then $\psi \in \mathrm{Hom}_{\mathcal{F}_P(G)}(N_\varphi, P)$ and $\psi|_Q = \varphi$. ☐

1.4. **Exotic fusion systems.** Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. If there is a finite group G with $P \in \mathrm{Syl}_p(G)$ such that $\mathcal{F} = \mathcal{F}_P(G)$, then we say that G *realizes* the fusion system $\mathcal{F}$. Surprisingly, not all saturated fusion systems can be realized by finite groups.

Definition 1.10. A saturated fusion system $\mathcal{F}$ is called *exotic* if there is no finite group G realizing it.

R. Solomon [24] discovered a family of exotic fusion systems in 1970s way before the notion of fusion systems were formalized while trying to characterize Conway's sporadic finite simple group Co_3 in terms of 2-fusion. Later brought back to attention by D. Benson [7] and formalized by R. Levi and B. Oliver [16][17], they are saturated fusion systems on Sylow 2-subgroups of the spin group $\mathrm{Spin}_7(q)$ for some odd prime power q, which are strictly bigger than the 2-fusion system of $\mathrm{Spin}_7(q)$. To this day, Solomon's exotic fusion systems are the only exotic fusion systems on finite 2-groups, and also the only exotic fusion systems proven to be exotic without using the classification of finite simple groups.

For an odd prime p, many exotic fusion systems have been discovered ever since. For example, A. Ruiz and A. Viruel [23] classified all saturated fusion systems on Sylow p-subgroups of $\mathrm{SL}_3(p)$, and found three exotic fusion systems when $p = 7$.

We'll see some more explicit examples of fusion systems in §2.4.

1.5. **Fusion systems of blocks of finite groups.** Blocks of finite groups also induce saturated fusion systems on their defect groups. In fact, fusion systems of finite groups can be viewed as a special case of fusion systems of blocks, as they are fusion systems of principal blocks. R. Kessar and R. Stancu [15] showed that Solomon's exotic fusion systems cannot be realized as fusion systems of some blocks. It is generalized believed that all fusion systems of blocks are fusion systems of some finite groups, but it has not been proven yet. We'll discuss fusion systems of blocks in more detail (including definitions of blocks and defect groups) in §4.1.

2. Local theory of fusion systems

2.1. **Local subsystems and Alperin's fusion theorem.** In group theory, normalizers and centralizers of nontrivial p-subgroups of a group are called *p-local subgroups.* One can define similar objects for fusion systems.

Definition 2.1. Let $\mathcal{F}$ be a fusion system on a finite p-group P and $Q \leq P$.

(1) Let $N_{\mathcal{F}}(Q)$ be the category whose object set is the set of all subgroups of $N_P(Q)$ and such that for all $U, V \leq N_P(Q)$,

$$\mathrm{Hom}_{N_{\mathcal{F}}(Q)}(U,V) = \{\varphi\colon U \to V \mid \exists \psi \in \mathrm{Hom}_{\mathcal{F}}(QU,QV)\colon \psi|_U = \varphi, \psi(Q) = Q\}$$

(2) Let $C_{\mathcal{F}}(Q)$ be the category whose object set is the set of all subgroups of $C_P(Q)$ and such that for all $U, V \leq C_P(Q)$,

$$\mathrm{Hom}_{C_{\mathcal{F}}(Q)}(U,V) = \{\varphi\colon U \to V \mid \exists \psi \in \mathrm{Hom}_{\mathcal{F}}(QU,QV)\colon \psi|_U = \varphi, \psi_Q = \mathrm{id}_Q\}$$

It is easy to see that $N_{\mathcal{F}}(Q)$ and $C_{\mathcal{F}}(Q)$ are fusion systems. But they are not necessarily saturated. The following theorem gives important special cases where $N_{\mathcal{F}}(Q)$ and $C_{\mathcal{F}}(Q)$ are saturated.

Theorem 2.2 (Puig; [18, 3.6])**.** *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P and $Q \leq P$.*

(1) *If Q is fully $\mathcal{F}$-normalized, then $N_{\mathcal{F}}(Q)$ is a saturated fusion system on $N_P(Q)$.*
(2) *If Q is fully $\mathcal{F}$-centralized, then $C_{\mathcal{F}}(Q)$ is a saturated fusion system on $C_P(Q)$.*

Example 2.3. Let G be a finite group with $P \in \mathrm{Syl}_p(G)$ and let $Q \leq P$.

(1) If Q is fully $\mathcal{F}_P(G)$-normalized, then $N_{\mathcal{F}_P(G)}(Q) = \mathcal{F}_{N_P(Q)}(N_G(Q))$.
(2) If Q is fully $\mathcal{F}_P(G)$-centralized, then $C_{\mathcal{F}_P(G)}(Q) = \mathcal{F}_{C_P(Q)}(C_G(Q))$.

For a fusion system $\mathcal{F}$ on a finite p-group P and a subgroup $Q \leq P$, we say that Q is *normal* in $\mathcal{F}$ if $\mathcal{F} = N_{\mathcal{F}}(Q)$; similarly, we say that Q is *central* in $\mathcal{F}$ if $\mathcal{F} = C_{\mathcal{F}}(Q)$.

Definition 2.4. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P.

(1) $O_p(\mathcal{F})$ is the subgroup of P generated by all subgroups of P which are normal in $\mathcal{F}$.
(2) $Z(\mathcal{F})$ is the subgroup of P generated by all subgroups of P which are central in $\mathcal{F}$.

One can easily see that $O_p(\mathcal{F})$ is the largest subgroup R of P such that $\mathcal{F} = N_{\mathcal{F}}(R)$ and that $Z(\mathcal{F})$ is the largest subgroup R of P such that $\mathcal{F} = C_{\mathcal{F}}(R)$.

Now we state and prove a fundamental theorem of Alperin, which says roughly that any saturated fusion system is generated by some of its normalizer subsystems.

Definition 2.5. Let $\mathcal{F}$ be a fusion system on a finite p-group P. A collection $\mathcal{C}$ of subgroups of P is called a *conjugation family* for $\mathcal{F}$ if the following condition is satisfied: for every isomorphism $\varphi\colon Q \to R$ in $\mathcal{F}$, there is $n \in \mathbb{N}$ and a sequence of $\mathcal{F}$-isomorphisms $\alpha_i \in \mathrm{Aut}_{\mathcal{F}}(U_i)$ $(1 \le i \le n)$ with $U_i \in \mathcal{C}$ such that

(1) $Q \le U_1$, $(\alpha_i \circ \cdots \circ \alpha_1)(Q) \le U_{i+1}$ for $1 \le i \le n-1$, and
(2) $\varphi = \alpha_n \circ \cdots \circ \alpha_1|_Q$.

Note that every conjugation family $\mathcal{C}$ for a fusion system $\mathcal{F}$ on a finite p-group P should contain P as its member.

Definition 2.6. Let $\mathcal{F}$ be a fusion system on a finite p-group P, and let $Q \le P$.

(1) Q is *$\mathcal{F}$-centric* if $C_P(Q') \le Q'$ for all $Q' \cong_{\mathcal{F}} Q$.
(2) Q is *$\mathcal{F}$-radical* if $O_p(\mathrm{Aut}_{\mathcal{F}}(Q)) = \mathrm{Aut}_Q(Q)$.

Theorem 2.7 (Alperin's fusion theorem)**.** *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. Then*

$$\mathcal{C} = \{U \le P \mid U \textit{ is fully } \mathcal{F}\textit{-normalized, } \mathcal{F}\textit{-centric and } \mathcal{F}\textit{-radical}\}$$

a conjugation family for $\mathcal{F}$.

We prove Alperin's fusion theorem by induction. For that, we use two lemmas—one from group theory and the other from the extension axiom of saturated fusion systems.

Lemma 2.8. *Let P be a finite p-group and $Q < P$. Then $Q < N_P(Q)$.*

Proof. Let $1 = Z_0 < Z_1 < \cdots < Z_n = P$ be the ascending central series for P. Here all the containments are proper because $Z(G) \ne 1$ if G is a finite p-group. Let i be maximal such that $Z_i \le Q$. Then $[Z_{i+1}, P] \le Z_i$, so $[Z_{i+1}, Q] \le Q$, so $Z_{i+1} \le N_P(Q)$. Since $Z_i < Z_{i+1}$, it follows that $Q < N_P(Q)$. □

Lemma 2.9. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. Let $\varphi\colon Q \to R$ be an isomorphism in $\mathcal{F}$ such that R is fully $\mathcal{F}$-normalized. Then there is $\sigma \in \mathrm{Aut}_{\mathcal{F}}(R)$ such that $\sigma \circ \varphi$ extends to an $\mathcal{F}$-morphism $\psi\colon N_P(Q) \to N_P(R)$.*

Proof. Since R is fully $\mathcal{F}$-normalized, $\mathrm{Aut}_P(R) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(R))$. On the other hand, $\varphi \circ \mathrm{Aut}_P(Q) \circ \varphi^{-1}$ is a p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(R)$. By Sylow's theorem, there is $\sigma \in \mathrm{Aut}_{\mathcal{F}}(R)$ such that $\sigma \circ \varphi \circ \mathrm{Aut}_P(Q) \circ \varphi^{-1} \circ \sigma^{-1} \le \mathrm{Aut}_P(R)$. This means that $N_{\sigma\circ\varphi} = N_P(Q)$. By the extension axiom, $\sigma\circ\varphi$ extends to an $\mathcal{F}$-morphism $\psi_1\colon N_P(Q) \to P$ and $\psi_1(N_P(Q)) \le N_P(R)$. By the axiom of fusion systems, ψ_1 restricts to an $\mathcal{F}$-morphism $\psi\colon N_P(Q) \to N_P(R)$. □

Proof of Theorem 2.7. First we show that the fully $\mathcal{F}$-normalized subgroups of P form a conjugacy family for $\mathcal{F}$. For this, we need to show that every $\mathcal{F}$-isomorphism $\varphi\colon Q \to R$ can be decomposed as in Definition 2.5. We proceed by induction on the index $|P : Q|$. First, if $Q = P$, then $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$ and P is trivially fully $\mathcal{F}$-normalized. So suppose that $Q < P$ and that every $\mathcal{F}$-isomorphism whose domain has order larger than $|Q|$ can be decomposed as in Definition 2.5. First consider the case where R is fully $\mathcal{F}$-normalized. By Lemma 2.9, there is $\sigma \in \mathrm{Aut}_{\mathcal{F}}(R)$ such that $\sigma \circ \varphi$ extends to $N_P(Q)$. Since $Q < P$, we have $Q < N_P(Q)$ by Lemma 2.8. Hence by assumption, $\sigma \circ \varphi$ decomposes as in Definition 2.5. Now $\varphi = \sigma^{-1} \circ (\sigma \circ \varphi)$ and $\sigma \in \mathrm{Aut}_{\mathcal{F}}(R)$ with R fully $\mathcal{F}$-normalized. Thus φ decomposes as desired. In general, take an $\mathcal{F}$-isomorphism $\psi\colon R \to S$ with S fully $\mathcal{F}$-normalized. Then

$\varphi = \psi^{-1} \circ (\psi \circ \varphi)$ and both $\psi \circ \varphi$ and ψ decomposes as in Definition 2.5 by the previous argument. Thus φ decomposes as desired.

Now if $Q \leq P$ is fully $\mathcal{F}$-normalized, but not $\mathcal{F}$-centric, then $C_P(Q) \not\leq Q$, so every $\mathcal{F}$-automorphism of Q extends to $QC_P(Q) > Q$. By induction on $|P : Q|$, we conclude that the fully $\mathcal{F}$-normalized and $\mathcal{F}$-centric subgroups of P form a conjugation family. Similarly, suppose that Q is fully $\mathcal{F}$-normalized and $\mathcal{F}$-centric, but not $\mathcal{F}$-radical. Then $\mathrm{Aut}_Q(Q) < O_p(\mathrm{Aut}_{\mathcal{F}}(Q)) \leq \mathrm{Aut}_P(Q)$. But $\mathrm{Aut}_Q(Q) \cong Q/Z(Q)$ and $\mathrm{Aut}_P(Q) \cong N_P(Q)/Z(Q)$. So there is $Q < R \leq N_P(Q)$ such that $O_p(\mathrm{Aut}_{\mathcal{F}}(Q)) \cong R/Z(Q)$, i.e. $O_p(\mathrm{Aut}_{\mathcal{F}}(Q)) = \mathrm{Aut}_R(Q)$. Since $O_p(\mathrm{Aut}_{\mathcal{F}}(Q)) = \mathrm{Aut}_R(Q)$ is normal in $\mathrm{Aut}_{\mathcal{F}}(Q)$, we have, for every $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$, $\varphi \circ \mathrm{Aut}_R(Q) \circ \varphi^{-1} = \mathrm{Aut}_R(Q)$, and so $N_\varphi \geq R$. By the extension axiom, φ extends to $R > Q$. By the same argument as above, we get that $\mathcal{C}$ is a conjugation family for $\mathcal{F}$. □

Let $\mathcal{F}$ be a fusion system on a finite p-group P, and let $Q \leq P$. A *fusion subsystem* of $\mathcal{F}$ on Q is a subcategory $\mathcal{E}$ of $\mathcal{F}$ which is a fusion system on Q. For $i = 1, 2$, let $\mathcal{E}_i$ be fusion subsystems of $\mathcal{F}$ on $Q_i \leq P$. Define $\mathcal{E}_1 \cap \mathcal{E}_2$ be the category whose object set is the set of all subgroups of $Q_1 \cap Q_2$ and such that for every $U, V \leq Q_1 \cap Q_2$,

$$\mathrm{Hom}_{\mathcal{E}_1 \cap \mathcal{E}_2}(U, V) = \mathrm{Hom}_{\mathcal{E}_1}(U, V) \cap \mathrm{Hom}_{\mathcal{E}_2}(U, V).$$

It is easy to check that $\mathcal{E}_1 \cap \mathcal{E}_2$ is also a fusion subsystem of $\mathcal{F}$. Hence we can define $\langle \mathcal{E}_1, \mathcal{E}_2 \rangle$ as the smallest fusion subsystem of $\mathcal{F}$ on $\langle Q_1, Q_2 \rangle$ containing $\mathcal{E}_1$ and $\mathcal{E}_2$. Then

Proposition 2.10. *Let $\mathcal{F}$ be a fusion system on a finite p-group P. Suppose that $\mathcal{C}$ is a conjugation family for $\mathcal{F}$. Then*

$$\mathcal{F} = \langle N_{\mathcal{F}}(U) \mid U \in \mathcal{C} \rangle.$$

That is, Alperin's fusion theorem tells us that every saturated fusion system is determined by its normalizer subsystems.

Remark 2.11. Let $\mathcal{F}$ be a fusion system on a finite p-group P. Even though $\mathcal{E}_i$ $(i = 1, 2)$ are saturated fusion subsystem of $\mathcal{F}$, the intersection fusion subsystem $\mathcal{E}_1 \cap \mathcal{E}_2$ is not necessarily saturated. In general, determining whether a fusion subsystem of a saturated fusion system is saturated is extremely difficult. See for example [8, 2.2].

Given Alperin's fusion theorem, Frobenius' normal p-complement theorem has a rather trivial analogue for fusion systems.

Proposition 2.12 (Frobenius)**.** *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let $\mathcal{C}$ be a conjugation family for $\mathcal{F}$. Then the following are equivalent.*

(1) $\mathcal{F} = \mathcal{F}_P(P)$.
(2) $N_{\mathcal{F}}(Q) = \mathcal{F}_{N_P(Q)}(N_P(Q))$ *for every* $Q \in \mathcal{C}$.
(3) $\mathrm{Aut}_{\mathcal{F}}(Q)$ *is a p-group for every* $Q \in \mathcal{C}$.

Proof. (1) ⇒ (2), (1) ⇒ (3): Clear.

(2) ⇒ (1), (3) ⇒ (1): Follows from Alperin's fusion theorem and that if Q is fully $\mathcal{F}$-normalized, then $\mathrm{Aut}_P(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$. □

2.2. **Constrained model theorem.** Now we state another fundamental theorem for fusion systems due to C. Broto, N. Castellana, J. Grodal, R. Levi and R. Oliver. Different from Alperin's fusion theorem, the proof of this theorem requires homological algebraic methods.

Definition 2.13. A saturated fusion system $\mathcal{F}$ on a finite p-group P is said to be *constrained* if there exists an $\mathcal{F}$-centric subgroup Q of P which is normal in $\mathcal{F}$.

Theorem 2.14 (Constrained model theorem; [8, 4.3])**.** *Let $\mathcal{F}$ be a constrained saturated fusion system on a finite p-group P. Then there is a unique (up to isomorphism) finite group G with $P \in \mathrm{Syl}_p(G)$ such that $\mathcal{F} = \mathcal{F}_P(G)$ and $C_G(O_p(G)) \leq O_p(G)$.*

Sketch of proof. The constrained condition implies that the cohomology groups

$$H^i(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_{\mathcal{F}}) \qquad (i > 0)$$

containing obstructions to the existence and uniqueness of a centric linking system associated with $\mathcal{F}$ vanish. Let $\mathcal{L}$ be the unique centric linking system associated with $\mathcal{F}$. Then $G := \mathrm{Aut}_{\mathcal{L}}(Q)$ does the job. See §4.2 for definitions and more details. □

We call the finite group G a *model* for $\mathcal{F}$.

Constrained model theorem has a slightly different (in fact equivalent) version. For this, we need to understand what $\mathcal{F}$-centric subgroups are for fusion systems of finite groups. Recall that for a fusion system $\mathcal{F}$ on a finite p-group P, $Q \leq P$ is $\mathcal{F}$-centric if and only if $C_P(Q') \leq Q'$ for every $Q' \cong_{\mathcal{F}} Q$. Note that $C_P(Q') \leq Q'$ is equivalent to $C_P(Q') = Z(Q')$.

Proposition 2.15. *Let G be a finite group with $P \in Syl_p(G)$, and let $Q \leq P$. Then Q is $\mathcal{F}_P(G)$-centric if and only if $Z(Q) \in \mathrm{Syl}_p(C_G(Q))$. Moreover, in this case, we have*

$$C_G(Q) = Z(Q) \times O_{p'}(C_G(Q)) = Z(Q) \times O_{p'}(N_G(Q)).$$

Proof. Write $\mathcal{F} = \mathcal{F}_P(G)$. Suppose that Q is $\mathcal{F}$-centric. Then Q is fully $\mathcal{F}$-centralized. Indeed, if $Q \cong_{\mathcal{F}} Q'$, then both Q and Q' are $\mathcal{F}$-centric, and $C_P(Q) = Z(Q) \cong Z(Q') = C_P(Q')$. Thus $Z(Q) = C_P(Q) \in \mathrm{Syl}_p(C_G(Q))$ by Proposition 1.8. Conversely, suppose that $Z(Q) \in \mathrm{Syl}_p(C_G(Q))$. Let $Q' \cong_{\mathcal{F}} Q$, i.e. $Q' = {}^xQ$ for some $x \in G$. Then conjugation by x gives $Z({}^xQ) \in \mathrm{Syl}_p(C_G({}^xQ))$. Since $C_P({}^xQ)$ is a p-subgroup of $C_G({}^xQ)$ containing $Z({}^xQ)$, it follows that $C_P({}^xQ) = Z({}^xQ)$. Thus Q is $\mathcal{F}$-centric.

Now $Z(Q) \in \mathrm{Syl}_p(C_G(Q))$ implies that $\mathcal{F}_{Z(Q)}(C_G(Q)) = \mathcal{F}_{Z(Q)}(Z(Q))$. So by Frobenius' normal p-complement theorem, there is $K \trianglelefteq C_G(Q)$ such that $C_G(Q) = Z(Q)K$ and $Z(Q) \cap K = 1$. Note that $Z(Q) \trianglelefteq C_G(Q)$. Then $[Z(Q), K] \leq Z(Q) \cap K = 1$. So $C_G(Q) = Z(Q) \times K$. Since $Z(Q) \in \mathrm{Syl}_p(C_G(Q))$, K consists of elements of $C_G(Q)$ of order prime to p. Thus $K = O_{p'}(C_G(Q))$.

Finally, we show that $O_{p'}(C_G(Q)) = O_{p'}(N_G(Q))$. Since $O_{p'}(C_G(Q))$ is characteristic in $C_G(Q)$ and $C_G(Q)$ is normal in $N_G(Q)$, $O_{p'}(C_G(Q))$ is normal in $N_G(Q)$. Thus $O_{p'}(C_G(Q)) \leq O_{p'}(N_G(Q))$. Conversely, note that $O_{p'}(N_G(Q))$ and Q normalize each other. Therefore, $[O_{p'}(N_G(Q)), Q] \leq O_{p'}(N_G(Q)) \cap Q = 1$. Thus $O_{p'}(N_G(Q)) \leq C_G(Q)$, and so $O_{p'}(N_G(Q)) \leq O_{p'}(C_G(Q))$. □

Proposition 2.16. *Let $\mathcal{F}$ be a constrained saturated fusion system on a finite p-group P. Fix an $\mathcal{F}$-centric subgroup Q of P which is normal in $\mathcal{F}$. Then there is a unique (up to isomorphism) finite group G with $P \in \mathrm{Syl}_p(G)$ such that $\mathcal{F} = \mathcal{F}_P(G)$, $Q \trianglelefteq G$, and $C_G(Q) \leq Q$.*

Proof. Let G be a model for $\mathcal{F}$. We show that $Q \leq G$ and $C_G(Q) \leq Q$. To show that $Q \trianglelefteq G$, write $Q_0 = O_p(G)$ and let $x \in G$. Then $Q_0 \trianglelefteq G$, so $c_x|_{Q_0} \in \mathrm{Aut}_{\mathcal{F}}(Q_0)$. Since Q is normal in $\mathcal{F}$, $c_x|_{Q_0} \in \mathrm{Aut}_{\mathcal{F}}(Q_0)$ extends to $c_y|_{QQ_0} \in \mathrm{Aut}_{\mathcal{F}}(QQ_0)$ for some $y \in G$. Then $y \in N_G(QQ_0)$

and $y^{-1}x \in C_G(Q) = Z(Q) \le N_G(QQ_0)$, so $x \in N_G(QQ_0)$. Thus $c_x|_{QQ_0} \in \mathrm{Aut}_{\mathcal{F}}(QQ_0)$. Since Q is normal in $\mathcal{F}$, we have $c_x(Q) = Q$, that is, $x \in N_G(Q)$. This shows that $Q \trianglelefteq G$. Now since Q is $\mathcal{F}$-centric, we have

$$C_G(Q) = Z(Q) \times O_{p'}(N_G(Q)) = Z(Q) \times O_{p'}(G)$$

by Proposition 2.15. But $O_{p'}(G) \le C_G(O_p(G)) \le O_p(G)$, so $O_{p'}(G) = 1$. Thus $C_G(Q) = Z(Q)$.

Conversely, suppose that H is a finite group with $P \in \mathrm{Syl}_p(H)$ such that $\mathcal{F} = \mathcal{F}_P(H)$, $Q \trianglelefteq H$, and $C_H(Q) \le Q$. Then $C_P(O_p(H)) \le C_P(Q) \le Q \le O_p(H)$. Thus H is a model for $\mathcal{F}$. By the uniqueness of model, we have $G \cong H$. □

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. For each $Q \le P$ which is fully $\mathcal{F}$-normalized and $\mathcal{F}$-centric, the normalizer subsystem $N_{\mathcal{F}}(Q)$ is saturated and constrained. Let $L_Q^{\mathcal{F}} = L_Q$ denote the unique finite group with $N_P(Q) \in \mathrm{Syl}(L_Q)$ such that $N_{\mathcal{F}}(Q) = \mathcal{F}_{N_P(Q)}(L_Q)$, $Q \trianglelefteq L_Q$, $C_{L_Q}(Q) \le Q$, given by the constrained model theorem. In particular, we have a short exact sequence of finite groups

$$1 \to Q \to L_Q \to \mathrm{Out}_{\mathcal{F}}(Q) \to 1,$$

where $\mathrm{Out}_{\mathcal{F}}(Q) := \mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Aut}_Q(Q)$.

We can extend the notion of the Sylow subgroup to infinite groups as follows. Let G be a group. If a finite p-subgroup P of G satisfies that every finite p-subgroup of G is G-conjugate to a subgroup of P, we say that P is a Sylow p-subgroup of G. If a fusion system $\mathcal{F}$ on a finite p-group P has a group G having P as a Sylow p-subgroup such that $\mathcal{F} = \mathcal{F}_P(G)$, we say that G *realizes* the fusion system $\mathcal{F}$. Combining those two fundamental theorems, G. Robinson showed that every saturated fusion system can be realized by some group.

Theorem 2.17 ([22, 2]). *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. There is a (possibly infinite) group G with $P \in \mathrm{Syl}_p(G)$ such that $\mathcal{F} = \mathcal{F}_P(G)$.*

Sketch of proof. Alperin's fusion theorem implies that $\mathcal{F}$ is generated, as a fusion system, by the normalizer subsystems $N_{\mathcal{F}}(Q)$ where Q runs over the conjugation family $\mathcal{C}$ consisting of the subgroups of P which are fully $\mathcal{F}$-normalized and $\mathcal{F}$-centric. List members of the family $\mathcal{C}$ as $Q_1, \ldots, Q_n$ and write $L_i = L_{Q_i}$ and $R_i = N_P(Q_i)$ for $1 \le i \le n$. We may assume that $Q_1 = R_1 = P$. Then the iterated amalgam

$$G := ((L_1 *_{R_2} L_2) *_{R_3} \cdots) *_{R_n} L_n$$

does the job. □

2.3. **Control of fusion.** Now we know that saturated fusion systems are determined by the normalizer subsystems. In finite group theory, this means that p-local subgroups determine the p-fusion pattern in a finite group. If there exists a single p-local subgroup H which determines the p-fusion pattern, we say that H *controls (p-)fusion* in G. In the fusion system setting, this amounts to the situation where $\mathcal{F} = N_{\mathcal{F}}(Q)$, i.e. there is a nontrivial subgroup Q of P which is normal in $\mathcal{F}$. Much work has been done on control of fusion in finite group theory, and recently many of them are generalized to fusion systems. We introduce some of the highlights here.

Proposition 2.18 (Burnside). *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. If P is abelian, then $\mathcal{F} = N_{\mathcal{F}}(P)$.*

Proof. Let $Q \leq P$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$. Since P is abelian, $N_P(\varphi(Q)) = P$, and so $\varphi(Q)$ is fully $\mathcal{F}$-normalized. By the extension axiom, φ extends to $QC_P(Q) = P$. Thus φ belongs to $N_{\mathcal{F}}(P)$. □

Combining the above theorem with Frobenius' theorem and the fact that the automorphism group of a finite cyclic 2-group is also a 2-group, we get an information on finite simple groups as follows.

Proposition 2.19. *Let G be a finite group with $1 \neq P \in \mathrm{Syl}_2(G)$. If P is cyclic, then G is not simple.*

Proof. Let $\mathcal{F} = \mathcal{F}_P(G)$ and suppose that P is cyclic. By Burnside's theorem, $\mathcal{F} = N_{\mathcal{F}}(P)$. This means that every $\mathcal{F}$-morphism is a restriction of an $\mathcal{F}$-automorphism of P. Since P is a finite cyclic 2-group, $\mathrm{Aut}(P)$ is also a 2-group. Hence $\mathrm{Aut}_{\mathcal{F}}(P)$ is a 2-group. Thus $\mathrm{Aut}_{\mathcal{F}}(Q)$ is a 2-group for all $Q \leq P$. By Frobenius' theorem, it follows that $\mathcal{F} = \mathcal{F}_P(P)$ and G has a normal p-complement, a contradiction. □

Note that Burnside's theorem says that if a finite p-group P is abelian, then P is normal in *any* saturated fusion system $\mathcal{F}$ on P. For more sophisticated results on control of fusion, we need to consider a certain characteristic subgroup of P introduced by J. Thompson.

Definition 2.20. Let P be a finite p-group. The *Thompson subgroup* $J(P)$ of P is the subgroup of P generated by all abelian subgroups of P of the maximal order.

In fact, what is going to play a special role is the center $Z(J(P))$ of the Thompson subgroup. Note that $Z(J(P))$ is a nontrivial characteristic subgroup of P. Hence if $\mathcal{F}$ is a saturated fusion system on P, then $N_{\mathcal{F}}(Z(J(P))$ is a saturated fusion subsystem of $\mathcal{F}$ on P. Using the Thompson subgroup, J. Thompson and G. Glauberman obtained some remarkable results on control of fusion in finite groups in 1960s. Later R. Kessar and M. Linckelmann generalized these results for saturated fusion systems.

Theorem 2.21 (Glauberman-Thompson normal p-complement theorem; [14, A])**.** *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P with p odd. Then $\mathcal{F} = \mathcal{F}_P(P)$ iff $N_{\mathcal{F}}(Z(J(P))) = \mathcal{F}_P(P)$.*

Definition 2.22. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P and let H be a finite group. We say that $\mathcal{F}$ is *H-free* if H is not involved in $L_Q^{\mathcal{F}}$ (i.e. $H \ncong K/L$ for any $L \trianglelefteq K \leq L_Q^{\mathcal{F}}$) for any $Q \leq P$ which is fully $\mathcal{F}$-normalized and $\mathcal{F}$-centric.

We write $Qd(p) = (C_p \times C_p) \rtimes \mathrm{SL}_2(p)$.

Theorem 2.23 (Glauberman's ZJ-Theorem; [14, B])**.** *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P with p odd. Suppose that $\mathcal{F}$ is $Qd(p)$-free. Then $\mathcal{F} = N_{\mathcal{F}}(ZJ(P))$.*

When generalizing group theoretic results to fusion systems, typically one uses the following reduction argument.

Sketch of generic proof. Suppose the theorem is false and take a counterexample $\mathcal{F}$ with minimal number of morphisms. First show that $O_p(\mathcal{F}) \neq 1$ in the following way. Suppose $O_p(\mathcal{F}) = 1$. It implies that for every $1 \neq Q \leq P$, $N_{\mathcal{F}}(Q) \neq \mathcal{F}$ because otherwise Q becomes a nontrivial subgroup of P which is normal in $\mathcal{F}$. Then, by induction, the theorem holds for $N_{\mathcal{F}}(Q)$ if $1 \neq Q \leq P$. Using this, together with Alperin's fusion theorem, show that the theorem holds for $\mathcal{F}$, which is a contradiction. This shows that $O_p(\mathcal{F}) \neq 1$. Then by further local analysis, show that $O_p(\mathcal{F})$ is $\mathcal{F}$-centric. Then constrained model theorem implies that $\mathcal{F} = \mathcal{F}_P(G)$ for some finite group G with $P \in \mathrm{Syl}_p(G)$. Use the original theorem involving finite groups to conclude that the theorem holds for $\mathcal{F}$, a contradiction. This finishes the proof. □

2.4. **Examples of fusion systems.** First we deal with the simplest case, namely when P is abelian.

Proposition 2.24. *If P is a finite abelian p-group, then every saturated fusion system $\mathcal{F}$ on P is realized by a finite group of the form $G = P \rtimes H$ for some finite p'-group H.*

Proof. Let $G = P \rtimes \mathrm{Aut}_{\mathcal{F}}(P)$. By the Sylow axiom, $\mathrm{Aut}_{\mathcal{F}}(P)$ is a finite p'-group, and so $P \in \mathrm{Syl}_p(G)$. By Burnside's theorem (Proposition 2.18), $\mathcal{F} = N_{\mathcal{F}}(P)$; that is, every $\mathcal{F}$-morphism is a restriction of some $\mathcal{F}$-automorphism of P. This implies that $\mathcal{F} = \mathcal{F}_P(G)$. □

For a nonabelian case, let us set $G := \mathrm{SL}_3(p)$ and consider saturated fusion systems on a Sylow p-subgroup of G. Since $|G| = p^3(p^2-1)(p^3-1)$, Sylow p-subgroups of G have order p^3. So

$$P := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in G \mid x, y, z \in \mathbb{F}_p \right\}.$$

is a Sylow p-subgroup of G. Let $\mathcal{F} = \mathcal{F}_P(G)$, and for $Q \leq P$, write

$$\mathrm{Out}_{\mathcal{F}}(Q) = \mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Aut}_Q(Q).$$

The normalizer $N_G(P)$ of P consists of all upper triangular matrices in G, and we have

$$\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_G(P)/\mathrm{Aut}_P(P) \cong N_G(P)/P \cong C_{p-1} \times C_{p-1}.$$

The Sylow P has exactly $p+1$ proper centric subgroups (i.e. subgroups $Q < P$ such that $C_P(Q) \leq Q$)

$$V_i = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & ix \\ 0 & 0 & 1 \end{pmatrix} \mid x, z \in \mathbb{F}_p \right\} (0 \leq i < p), \quad V_p = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid y, z \in \mathbb{F}_p \right\},$$

and they are all elementary abelian groups of order p^2, except for $V_1 \cong C_4$ when $p = 2$. Exactly two of them are both $\mathcal{F}$-centric and $\mathcal{F}$-radical, namely,

$$V_0 = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G \right\}, \quad V_p = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \in G \right\},$$

with normalizers

$$N_G(V_0) = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in G \right\}, \quad N_G(V_p) = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\},$$

respectively. Hence

$$\mathrm{Out}_{\mathcal{F}}(V_0) \cong N_G(V_0)/V_0 \cong \mathrm{GL}_2(p), \quad \mathrm{Out}_{\mathcal{F}}(V_p) \cong N_G(V_p)/V_p \cong \mathrm{GL}_2(p).$$

By Alperin's fusion theorem, the above data fully describes the fusion system $\mathcal{F} = \mathcal{F}_P(G)$. We summarize it as follows.

$\mathcal{F}^{fcr}$	Out	Group
P; V_0; V_p	$(p-1)^2$; $\mathrm{GL}_2(p)$; $\mathrm{GL}_2(p)$	$\mathrm{SL}_3(p)$

Now when $p = 2$, P is a dihedral group of order 8. Note that $\mathrm{Out}_{\mathcal{F}}(P) = 1$ and $\mathrm{GL}_2(2) \cong S_3$ in this case. There are two other saturated fusion systems on P, up to isomorphism. One of them has

$$\mathrm{Out}_{\mathcal{F}}(V_0) \cong S_3, \qquad \mathrm{Out}_{\mathcal{F}}(V_2) = 1,$$

and it is realized by S_4 on $P = \langle (1,2,3,4), (12)(34) \rangle$. The other has

$$\mathrm{Out}_{\mathcal{F}}(V_0) = \mathrm{Out}_{\mathcal{F}}(V_2) = 1,$$

and by Frobenius' theorem (Proposition 2.12), it is realized by P itself. In particular, there is no exotic fusion system on the dihedral group of order 8.

$\mathcal{F}^{fcr}$	Out	Group
P; V_0; V_p	1; S_3; S_3	$\mathrm{GL}_3(2)$
P; V_0	1; S_3	S_4
P	1	D_8

Things are more interesting when p is odd. A. Ruiz and A. Viruel classified all saturated fusion systems on P, and roughly they come in two families: one of them consists of fusion systems of extensions of $\mathrm{PSL}_3(p)$ (and some other smaller fusion systems); the other consists of fusion systems of some sporadic finite simple groups and a Tits group at $p = 3, 5, 7, 13$ (including the Monster group M at $p = 13$) and three exotic fusion systems at $p = 7$ which can be viewed as enlargements of fusion systems of some sporadic simple groups in this family. We list two saturated fusion systems on P when $p = 7$ as follows.

$\mathcal{F}^{fcr}$	Out	Group
P; $V_1, \ldots, V_6$	$6^2\!:\!2$; $\mathrm{SL}_2(7)\!:\!2$	Fi_{24}
P; $V_1, \ldots, V_6$; V_0, V_7	$6^2\!:\!2$; $\mathrm{SL}_2(7)\!:\!2$; $\mathrm{GL}_2(7)$	

Here $\mathcal{F}$-conjugacy classes are distinguished by semicolons. The second row represents the exotic 7-fusion system which contains the 7-fusion system of the Fischer group Fi_{24}.

3. Structure theory of fusion systems

3.1. **Normal subsystems and simple fusion systems.** By structure theory, we mean breaking down given saturated fusion systems into 'simple' pieces. We start with some relevant definitions.

Definition 3.1. Let $\mathcal{F}$ be a fusion system on a finite p-group S and let $Q \leq P$.

(1) Q is *weakly $\mathcal{F}$-closed* if for every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ we have $\varphi(Q) = Q$.
(2) Q is *strongly $\mathcal{F}$-closed* if for every $R \leq Q$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, P)$ we have $\varphi(R) \subseteq Q$.

If $\mathcal{F} = N_{\mathcal{F}}(Q)$, then Q is strongly $\mathcal{F}$-closed. Clearly, if $Q \leq P$ is strongly $\mathcal{F}$-closed, then Q is weakly $\mathcal{F}$-closed; if Q is weakly $\mathcal{F}$-closed, then $Q \trianglelefteq P$.

Definition 3.2. Let $\mathcal{F}$ be a fusion system on a finite p-group P. Let $\mathcal{E}$ be a fusion subsystem of $\mathcal{F}$ on $Q \leq P$. We say that $\mathcal{E}$ is *$\mathcal{F}$-invariant* in $\mathcal{F}$ if the following conditions are satisfied.

(1) Q is strongly $\mathcal{F}$-closed.

(2) Whenever $R \leq Q$, $\alpha \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$, $S \leq R$, $\varphi \in \mathrm{Hom}_{\mathcal{E}}(S, Q)$, we have $\alpha \circ \varphi \circ \alpha^{-1} \in \mathrm{Hom}_{\mathcal{E}}(\alpha(S), Q)$.

Definition 3.3. Let $\mathcal{F}$ be a fusion system on a finite p-group P. Let $\mathcal{E}$ be a fusion subsystem of $\mathcal{F}$ on $Q \leq P$. We say that $\mathcal{E}$ is *$\mathcal{F}$-Frattini* in $\mathcal{F}$ if the following conditions are satisfied.

(1) Q is strongly $\mathcal{F}$-closed.
(2) For every $R \leq Q$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$, there exist $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q)$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(\alpha(R), Q)$ such that $\varphi = \psi \circ \alpha|_R$.

The terminology $\mathcal{F}$-Frattini comes from the famous Frattini argument in group theory.

Lemma 3.4 (Frattini argument)**.** *Let G be a finite group. Suppose that K is a normal subgroup of G and P is a Sylow p-subgroup of K. Then we have $G = KN_G(P)$.*

Proof. Let $x \in G$. Then ${}^xP \leq {}^xK = K$ because K is normal in G, and so ${}^xP$ is a Sylow p-subgroup of K. By Sylow's theorem, there exists $y \in K$ such that ${}^xP = {}^yP$, that is, $y^{-1}x \in N_G(P)$. □

The following theorem, due to M. Aschbacher, shows that an analogue of Frattini argument holds for saturated fusion systems, and moreover, it almost characterizes invariant subsystems.

Proposition 3.5 ([5, 3.3])**.** *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. Let $\mathcal{E}$ be a fusion subsystem of $\mathcal{F}$ on $Q \leq P$. Then $\mathcal{E}$ is $\mathcal{F}$-invariant if and only if $\mathcal{E}$ is $\mathcal{F}$-Frattini and $\mathrm{Aut}_{\mathcal{F}}(Q) \leq \mathrm{Aut}(\mathcal{E})$.*

Proof. Suppose that $\mathcal{E}$ is $\mathcal{F}$-Frattini and $\mathrm{Aut}_{\mathcal{F}}(Q) \leq \mathrm{Aut}(\mathcal{E})$. Let $R \leq Q$, $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$. Since $\mathcal{E}$ is $\mathcal{F}$-Frattini, $\varphi = \psi \circ \alpha$ for some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q)$ and $\psi \in \mathrm{Hom}_{\mathcal{E}}(\alpha(R), Q)$. Also $\mathrm{Aut}_{\mathcal{F}}(Q) \leq \mathrm{Aut}(\mathcal{E})$. Thus φ sends $\mathcal{E}$-maps inside R to $\mathcal{E}$-maps inside $\varphi(R)$.

Conversely, suppose that $\mathcal{F}$-invariant. Clearly $\mathrm{Aut}_{\mathcal{F}}(Q) \leq \mathrm{Aut}(\mathcal{E})$. To show that $\mathcal{E}$ is $\mathcal{F}$-Frattini, let $R \leq Q$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$. We argue by induction on $|P : R|$. The case $R = P$ being trivial, assume $R < P$. By Alperin's fusion theorem, we have a decomposition

$$\varphi = \beta_n \circ \cdots \circ \beta_1|_R, \quad \beta_i \in \mathrm{Aut}_{\mathcal{F}}(U_i), \quad U_i \in \mathcal{F}^{fcr}.$$

Suppose that $\beta_i = \psi_i \circ \alpha_i$ for some $\alpha_i \in \mathrm{Aut}_{\mathcal{F}}(U_i)$ and $\psi \in \mathrm{Aut}_{\mathcal{E}}(U_i)$ for every i. Then

$$\varphi = (\psi_n \circ \alpha_n \psi_{n-1} \alpha_n^{-1} \circ \cdots) \circ (\alpha_n \circ \cdots \circ \alpha_1)|_R$$

gives a desired decomposition. So we may assume that $\varphi \in \mathrm{Aut}_{\mathcal{F}}(R)$ with $R \in \mathcal{F}^{fcr}$. We show that φ 'almost' extends to a subgroup of P strictly larger than R by using that $\mathcal{E}$ is $\mathcal{F}$-invariant and Frattini argument. Then the proof will be finished by induction.

Since $\mathcal{E}$ is $\mathcal{F}$-invariant, we have $\mathrm{Aut}_{\mathcal{E}}(R) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(R)$. Since R is fully $\mathcal{F}$-normalized, we have $\mathrm{Aut}_P(R) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(R))$. Then $S := \mathrm{Aut}_P(R) \cap \mathrm{Aut}_{\mathcal{E}}(R) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{E}}(R))$. Set $T = \{x \in N_P(R) \mid c_x|_R \in \mathrm{Aut}_{\mathcal{E}}(R)\}$. Then $S = \mathrm{Aut}_T(R)$. By Frattini argument, we have

$$\mathrm{Aut}_{\mathcal{F}}(R) = \mathrm{Aut}_{\mathcal{E}}(R) N_{\mathrm{Aut}_{\mathcal{F}}(R)}(\mathrm{Aut}_T(R)).$$

Then $\varphi = \psi_1 \circ \varphi_1$ for some $\varphi_1 \in N_{\mathrm{Aut}_{\mathcal{F}}(R)}(\mathrm{Aut}_T(R))$ and $\psi_1 \in \mathrm{Aut}_{\mathcal{E}}(R)$. By definition of N_{φ_1}, we have $N_{\varphi_1} \geq T$. Since $\mathcal{E}$ is defined on Q and $R < Q$, we have $T \geq N_Q(R) > R$. Since R is fully $\mathcal{F}$-normalized, the extension axiom implies that φ_1 extends to $T > R$. By induction, $\varphi_1 = \psi_2 \circ \alpha$ for some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q)$ and $\psi_2 \in \mathrm{Aut}_{\mathcal{E}}(R)$. Thus $\varphi = \psi_1 \circ \varphi_1 = \psi_1 \circ \psi_2 \circ \alpha$. Since $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q)$ and $\psi_1 \circ \varphi_1 \in \mathrm{Aut}_{\mathcal{E}}(Q)$, we are done. □

Definition 3.6. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. A fusion subsystem $\mathcal{E}$ of $\mathcal{F}$ on $Q \leq P$ is called *normal* in $\mathcal{F}$ if it is saturated and $\mathcal{F}$-invariant.

Definition 3.7. A saturated fusion system $\mathcal{F}$ on a finite p-group P is said to be *simple* if it has no normal subsystem other than $\mathcal{F}$ itself and the trivial subsystem on $\{1\}$.

Even though this definition of normal subsystems seems to be natural, Aschbacher has proposed a slightly stronger notion of normal subsystems (and hence a weaker notion of simple fusion systems). Following D. Craven's terminology in [13], we call Aschbacher's notion of normality *strong normality.*

Definition 3.8 (Aschbacher)**.** Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. A fusion subsystem $\mathcal{E}$ of $\mathcal{F}$ on $Q \leq P$ is called *strongly normal* in $\mathcal{F}$ if it is a normal subsystem of $\mathcal{F}$ and

every $\varphi \in \mathrm{Aut}_{\mathcal{E}}(Q)$ extends to $\psi \in \mathrm{Aut}_{\mathcal{F}}(QC_P(Q))$ such that $[\psi, C_P(Q)] \leq Z(Q)$.

The extra condition in the previous definition is called the (N1) *property.*

The reason why Aschbacher prefers strong normality is that strongly normality seems to reflect finite group world better than normality does. Here we give two such examples. First, of course, fusion systems of finite groups satisfy the (N1) property. Note that the proof of the following proposition appearing in Aschbacher's orginal paper is slightly misstated. We thank R. Kessar for providing this version of a proof.

Proposition 3.9 ([5, 6.3])**.** *Let G be a finite group with Sylow p-subgroup P. Let H be a normal subgroup of G. Then $\mathcal{F}_{P\cap H}(H)$ is a strongly normal subsystem of $\mathcal{F}_P(G)$.*

Proof. Set $Q = P \cap H$. Then $Q \in \mathrm{Syl}_p(H)$, so $\mathcal{F}_Q(H)$ is saturated. Clearly $\mathrm{Aut}_G(Q) \leq \mathrm{Aut}(\mathcal{F}_Q(H))$ because $H \trianglelefteq G$. To show that $\mathcal{F}_Q(H)$ is $\mathcal{F}$-Frattini, let $R \leq Q$, $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, P)$. Then $\varphi = c_x|_R$ for some $x \in G$. Then ${}^xR \leq P$. Since $R \leq Q = P \cap H$ and $H \trianglelefteq G$, we also have ${}^xR \leq H$. Thus ${}^xR \leq P \cap H = Q$. This shows that R is strongly $\mathcal{F}$-closed. By Frattini argument, we have $G = HN_G(Q)$. So $x = yz$ for some $y \in H$ and $z \in N_G(Q)$. Let $\alpha = c_z|_Q \in \mathrm{Aut}_G(Q)$, $\psi = c_y|_{\alpha(R)} \in \mathrm{Hom}_H(\alpha(R), Q)$. Then $\varphi = \psi \circ \alpha|_R$, which shows that $\mathcal{F}_Q(H)$ is $\mathcal{F}_P(H)$-Frattini.

Now we show that $\mathcal{F}_Q(H)$ satisfies (N1) property. For this purpose, we may restrict our attention to the group $H_1 := HC_P(Q)$. It contains H as a normal subgroup, and $Q_1 := QC_P(Q)$ as a Sylow p-subgroup. Thus $\mathcal{F}_{Q_1}(H_1)$ contains $\mathcal{F}_Q(H)$ as a saturated invariant subsystem by the previous paragraph.

Suppose $\varphi \in \mathrm{Aut}_H(Q)$. Set $K = N_{H_1}(Q) \cap N_{H_1}(Q_1)$. By extension axiom, there is $g \in K$ such that $\varphi = c_g|_Q$. Now $C_K(Q) \trianglelefteq K$ and $Q_1 \in \mathrm{Syl}_p(K)$. So $C_K(Q) \cap Q_1 = C_P(Q) \in \mathrm{Syl}_p(C_K(Q))$. Then by Frattini argument, we have

$$K = C_K(Q)N_K(C_P(Q)).$$

Thus we may assume that $g \in N_K(C_P(Q))$. Moreover, $K = (N_H(Q) \cap N_H(Q_1))C_P(Q)$. So we may assume further that $g \in N_H(Q) \cap N_H(C_P(Q))$. Then for every $u \in C_P(Q)$, $[g, u] \in H \cap C_P(Q) = Z(Q)$ because H is normal in G and g normalizes $C_P(Q)$. ☐

Then comes an important application of strong normality.

Theorem 3.10 ([5, 1])**.** *Let $\mathcal{F}$ be a saturated constrained fusion system on a finite p-group P, and let G be a model for $\mathcal{F}$. Then there is a one-to-one correspondence between normal subgroups of G and strongly normal subsystems of $\mathcal{F}$.*

Using this theorem and others, Aschbacher has been able to re-establish some fundamental objects and theorems of finite group theory for fusion systems.(See [2], [3], [4]) Even though a lot of work has to be done, it is hoped that strategies used for the classification of finite simple groups can be suitably modified to classify all saturated fusion systems on finite 2-groups.

3.2. **Quotient systems.** to be updated

3.3. $O^p(\mathcal{F})$ **and** $O^{p'}(\mathcal{F})$**.** to be updated

3.4. p**-Solvable fusion systems.** to be updated

4. Applications

4.1. **Block theory: fusion systems of blocks of finite groups.** to be updated

4.2. p**-Local homotopy theory:** p**-local finite groups.** There is a functor on the category Top of topological spaces

$$(-)^\wedge_p \colon \mathsf{Top} \to \mathsf{Top},$$

called the *Bousfield-Kan p-completion functor*, whose main property is that a map $f\colon X \to Y$ of topological spaces induces a homotopy equivalence $f^\wedge_p\colon X^\wedge_p \to Y^\wedge_p$ if and only if f induces an isomorphism $f^*\colon H^*(Y, \mathbb{F}_p) \to H^*(X, \mathbb{F}_p)$. We say that X and Y have the same *p-local homotopy type*, or that they are *mod p equivalent*, if $X^\wedge_p \simeq Y^\wedge_p$, i.e. if $X^\wedge_p$ is homotopy equivalent to $Y^\wedge_p$

Theorem 1.3 of Cartan and Eilenberg indicates that the p-fusion system of a finite group G is related to the p-local homotopy type of its classifying space BG. This connection is made more precise in the following theorem proven by R. Oliver.

Theorem 4.1 (Martino-Priddy Conjecture; [19],[20])**.** *Let G, G' be finite groups and p a prime. The following are equivalent.*

1. $BG^\wedge_p \simeq BG'^\wedge_p$.
2. *There is a group isomorphism $\varphi\colon S \to S'$ from $S \in \mathrm{Syl}_p(G)$ to $S' \in \mathrm{Syl}_p(G')$ inducing an equivalence of categories $\mathcal{F}_S(G) \cong \mathcal{F}_{S'}(G')$.*

In other words, the p-local homotopy type of BG is completely determined by the fusion system $\mathcal{F}_S(G)$ and vice versa. In this section, we sketch how the proof of this theorem goes. First we 'extend' saturated fusion systems to centric linking systems. For a fusion system $\mathcal{F}$ on a finite p-group S, let $\mathcal{F}^c$ denote the full subcategory of $\mathcal{F}$ consisting of $\mathcal{F}$-centric subgroups of S.

Definition 4.2. Let $\mathcal{F}$ be a fusion system on a finite p-group S. A *centric linking system* associated with $\mathcal{F}$ is a category $\mathcal{L}$ whose object set is the set of all $\mathcal{F}$-centric subgroups of S, together with a functor

$$\pi\colon \mathcal{L} \to \mathcal{F}^c$$

and injective group homomorphisms

$$\delta_P\colon P \to \mathrm{Aut}_{\mathcal{L}}(P)$$

for each $\mathcal{F}$-centric subgroup P of S, which satisfies the following conditions.

(1) The functor π is the identity on objects and surjective on morphisms. For each pair P, Q of $\mathcal{F}$-centric subgroups of S, $Z(P)$ acts freely on $\mathrm{Hom}_{\mathcal{L}}(P,Q)$ by composition (upon identifying $Z(P)$ with $\delta(Z(P)) \leq \mathrm{Aut}_{\mathcal{L}}(P)$), and π induces a bijection

$$\mathrm{Hom}_{\mathcal{L}}(P,Q)/Z(P) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{F}}(P,Q).$$

(2) For each $\mathcal{F}$-centric subgroup P of S and for each $x \in P$, we have

$$\pi(\delta_P(x)) = c_x|_P.$$

(3) For each pair P, Q of $\mathcal{F}$-centric subgroups of S, each $f \in \mathrm{Hom}_{\mathcal{L}}(P,Q)$, and $x \in P$, the following square commutes.

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ {\scriptstyle \delta_P(x)}\downarrow & & \downarrow{\scriptstyle \delta_Q(\pi(f)(x))} \\ P & \xrightarrow{f} & Q \end{array}$$

Definition 4.3. A *p-local finite group* is a triple $(S, \mathcal{F}, \mathcal{L})$ consisting of a finite p-group S, a saturated fusion system $\mathcal{F}$ on S, and a centric linking system $\mathcal{L}$ associated with $\mathcal{F}$. The *classifying space* of a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is $|\mathcal{L}|^{\wedge}_p$, where $|\mathcal{L}|$ denotes the (geometric realization of) nerve of the category $\mathcal{L}$.

In general, it is an open question whether a saturated fusion system $\mathcal{F}$ has a centric linking system $\mathcal{L}$, or whether such $\mathcal{L}$ is unique. But in case of a fusion system of a finite group, one can easily construct a centric linking system associate with it.

Let G be a finite group with $S \in \mathrm{Syl}_p(G)$. Then $\mathcal{F}_S(G)$ is a saturated fusion system on S. For $P, Q \leq S$, let

$$N_G(P,Q) = \{x \in G \mid {}^xP \leq Q\}.$$

In particular, $N_G(P,P) = N_G(P)$. Then we have

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q) \cong N_G(P,Q)/C_G(P)$$

where the bijection is induced by the map sending $x \in N_G(P,Q)$ to $c_x \colon P \to Q$. In other words, a morphism in $\mathcal{F}_S(G)$ from P to Q can be represented by a coset $xC_G(P)$ for some $x \in N_G(P,Q)$. If $R \leq S$ and $yC_G(Q)$ with $y \in N_G(Q,R)$ represent a morphism in $\mathcal{F}_S(G)$ from Q to R, then their composition is represented by

$$yC_G(Q)xC_G(P) = yxx^{-1}C_G(Q)xC_G(P) = yxC_G(x^{-1}Qx)C_G(P) = yxC_G(P).$$

Note that the last equality holds because $x \in N_G(P,Q)$ implies that $x^{-1}Qx \geq P$ and hence $C_G(x^{-1}Qx) \leq C_G(P)$.

For an $\mathcal{F}_S(G)$-centric subgroup P of S, let us write $C'_G(P) := O_{p'}(C_G(P))$ for short. By Proposition 2.15, we have

$$C_G(P) = Z(P) \times C'_G(P).$$

Now define $\mathcal{L}^c_S(G)$ to be the category whose object set is the set of all $\mathcal{F}$-centric subgroups of S and such that

$$\mathrm{Hom}_{\mathcal{L}^c_S(G)}(P,Q) = N_G(P,Q)/C'_G(P)$$

for all $\mathcal{F}$-centric subgroups P, Q of S, with composition of morphisms given as follows: for $\mathcal{F}$-centric subgroups $P, Q, R \leq S$ and $x \in N_G(P,Q)$, $y \in N_G(Q,R)$, the composition of $xC'_G(P)$ and $yC'_G(Q)$ is given by

$$yC'_G(Q)xC'_G(P) = yxx^{-1}C'_G(Q)xC'_G(P) = yxC'_G(x^{-1}Qx)C'_G(P) = yxC'_G(P).$$

We get the second equality because $C'_G(Q) = O_{p'}(C_G(Q))$ is a characteristic subgroup of $C_G(Q)$. For the third equality, observe that $C'_G(P)$ is the set of all elements of $C_G(P)$ of order prime to p, and hence $C'_G(x^{-1}Qx) \leq C'_G(P)$.

Proposition 4.4. *Let G be a finite group with $S \in \mathrm{Syl}_p(G)$. Then the triple*

$$(S, \mathcal{F}_S(G), \mathcal{L}^c_S(G))$$

is a p-local finite group.

Proof. We've already seen that $\mathcal{F}_S(G)$ is a saturated fusion system on S in Proposition 1.9. So it remains to show that $\mathcal{L}^c_S(G)$ is a centric linking system associated with $\mathcal{F}_S(G)$. Define $\pi\colon \mathcal{L}^c_S(G) \to \mathcal{F}_S(G)^c$ to be the functor which is the identity on objects and such that for each pair P, Q of $\mathcal{F}$-centric subgroups of S,

$$\begin{array}{rccc} \pi_{P,Q}\colon & N_G(P,Q)/C'_G(P) & \to & N_G(P,Q)/C_G(P) \\ & xC'_G(P) & \mapsto & xC_G(P) \end{array}$$

For each $\mathcal{F}$-centric subgroup P of S, define

$$\begin{array}{rcccl} \delta_P\colon & P & \to & N_G(P)/C'_G(P) & = \mathrm{Aut}_{\mathcal{L}^c_S(G)}(P) \\ & x & \mapsto & xC'_G(P). & \end{array}$$

One can easily check that the three conditions in Definition 4.2 are satisfied. □

Now we state two propositions about p-local finite groups and their classifying spaces without proof.

Proposition 4.5 ([9, 1.1])**.** *Let G be a finite group with $S \in \mathrm{Syl}_p(G)$. Then $|\mathcal{L}^c_S(G)|^\wedge_p \simeq BG^\wedge_p$.*

We say that two p-local finite groups $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ are *isomorphic* if they are isomorphic as triples, via isomorphisms of groups and categories which commute with all the structures which link them.

Proposition 4.6 ([10, 7.4])**.** *Two p-local finite groups $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ are isomorphic if and only if $|\mathcal{L}|^\wedge_p \simeq |\mathcal{L}'|^\wedge_p$.*

These two propositions together show the implication (1) ⇒ (2) in Martino-Priddy conjecture. Moreover, the other implication (2) ⇒ (1) is equivalent to the uniqueness of centric linking system for $\mathcal{F}_S(G)$.

Theorem 4.7 ([19],[20])**.** *Let G be a finite group with Sylow p-subgroup S. Then $\mathcal{L}^c_S(G)$ is the unique centric linking system associated with $\mathcal{F}_S(G)$.*

To prove this theorem, Oliver identified a general condition for the existence and uniqueness of centric linking systems. This condition is about vanishing of certain functor cohomology.

Let $\mathcal{F}$ be a fusion system on a finite p-groups S. Let $\mathcal{F}^c$ be the full subcategory of $\mathcal{F}$ consisting of $\mathcal{F}$-centric subgroups of S. We define the *orbit category* $\mathcal{O}(\mathcal{F}^c)$ of $\mathcal{F}^c$ as the category whose object set is the set of all $\mathcal{F}$-centric subgroups of S and such that for every $\mathcal{F}$-centric $P, Q \leq S$,

$$\mathrm{Hom}_{\mathcal{O}(\mathcal{F}^c)}(P,Q) = \mathrm{Inn}(Q)\backslash \mathrm{Hom}_{\mathcal{F}^c}(P,Q).$$

Here $\mathrm{Inn}(Q) = \mathrm{Aut}_Q(Q)$. Namely, a morphism in $\mathrm{Hom}_{\mathcal{O}(\mathcal{F}^c)}(P,Q)$ is an $\mathrm{Inn}(Q)$-orbit of $\mathcal{F}$-morphisms $[\varphi]$ for $\varphi \in \mathrm{Hom}_{\mathcal{F}^c}(P,Q)$ such that if $\psi \in \mathrm{Hom}_{\mathcal{F}^c}(P,Q)$, $[\varphi] = [\psi]$ if and only if $\psi = c_x \circ \varphi$ for some $x \in Q$. If $\varphi \in \mathrm{Hom}_{\mathcal{F}^c}(P,Q)$ and $\psi \in \mathrm{Hom}_{\mathcal{F}^c}(Q,R)$, the composition is defined by $[\psi] \circ [\varphi] = [\psi \circ \varphi]$. It is easy to check that $\mathcal{O}(\mathcal{F}^c)$ is a well-defined category.

Suppose that P, Q are $\mathcal{F}$-centric subgroups and $\varphi\colon P \to Q$ is an $\mathcal{F}$-morphism. Then

$$Z(Q) = C_S(Q) \leq C_S(\varphi(P)) = Z(\varphi(P)) = \varphi(Z(P)).$$

So we have a map $\varphi^{-1}\colon Z(Q) \to Z(P)$. Thus we have a contravariant functor

$$\mathcal{F}^c \to \mathsf{Ab}$$

where Ab denotes the category of abelian groups given by sending each $\mathcal{F}$-centric subgroup P of S to its center $Z(P)$ and sending each $\varphi \in \mathrm{Hom}_{\mathcal{F}^c}(P,Q)$ to $\varphi^{-1}\colon Z(Q) \to Z(P)$. Furthermore, one can easily check that this functor induces a functor $\mathcal{O}(\mathcal{F}^c) \to \mathsf{Ab}$, which we denote by $\mathcal{Z}_{\mathcal{F}}$. Namely,

$$\mathcal{Z}_{\mathcal{F}}\colon \mathcal{O}(\mathcal{F}^c) \to \mathsf{Ab}$$

is the functor sending each $\mathcal{F}$-centric subgroup P of S to $Z(P)$ and each $[\varphi]$ with $\varphi \in \mathrm{Hom}_{\mathcal{F}^c}(P,Q)$ to $\varphi^{-1}\colon Z(Q) \to Z(P)$.

Proposition 4.8. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S.*

(1) *The obstruction to the existence of a centric linking system associated with $\mathcal{F}$ lies in*

$$H^3(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_{\mathcal{F}}).$$

(2) *If there exists a centric linking system $\mathcal{L}$ associated with $\mathcal{F}$, this $\mathcal{L}$ is unique if and only if*

$$H^2(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_{\mathcal{F}}) = 0.$$

Now comes the main content of Martino-Priddy conjecture.

Theorem 4.9 ([19],[20]). *Let G be a finite group with Sylow p-subgroup S. Let $\mathcal{F} = \mathcal{F}_S(G)$. Then*

$$H^i(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_{\mathcal{F}}) = 0 \quad \text{for all } i \geq 2.$$

The proof of this theorem runs more than 130 pages and uses the classification of finite simple groups.

Now Theorem 4.7 follows as a corollary, and hence Martino-Priddy conjecture.

Appendix A. Transfer theory

to be updated

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