The average size of the kernel of a matrix and orbits of linear groups

Tobias Rossmann

Department of Mathematics
University of Auckland
Auckland
New Zealand

Current address:
School of Mathematics, Statistics and Applied Mathematics
National University of Ireland, Galway
Galway
Ireland

Let $\mathfrak{O}$ be a compact discrete valuation ring of characteristic zero. Given a module M of matrices over $\mathfrak{O}$, we study the generating function encoding the average sizes of the kernels of the elements of M over finite quotients of $\mathfrak{O}$. We prove rationality and establish fundamental properties of these generating functions and determine them explicitly for various natural families of modules M. Using p-adic Lie theory, we then show that special cases of these generating functions enumerate orbits and conjugacy classes of suitable linear pro-p groups.

Contents

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1 Introduction

This article is devoted to certain generating functions $\mathsf{Z}^{\mathsf{ask}}_M(T)$ ("ask zeta functions") attached to modules M of matrices over compact discrete valuation rings. The coefficients of $\mathsf{Z}^{\mathsf{ask}}_M(T)$ encode the average sizes of the kernels of the elements of M over finite quotients of the base ring.

Prior to formally defining these functions and stating our main results, we briefly indicate how our study of the functions $\mathsf{Z}^{\mathsf{ask}}_M(T)$ is motivated by questions from (both finite and infinite) group theory and probabilistic linear algebra.

Conjugacy classes of finite groups. Given a finite group G, let $\mathrm{k}(G)$ denote the number of its conjugacy classes. It is well-known that $\mathrm{k}(G)$ coincides with the number of the (ordinary) irreducible characters of G. Let $\mathrm{U}_d \leqslant \mathrm{GL}_d$ be the group scheme of upper unitriangular $d \times d$ matrices. Raised as a question in [50], "Higman's conjecture" asserts that $\mathrm{k}(\mathrm{U}_d(\mathbf{F}_q))$ is given by a polynomial in q for each fixed $d \geqslant 1$.

Numerous people have contributed to confirming Higman's conjecture for small d. In particular, building on a long series of papers, Vera-López and Arregi [87] established Higman's conjecture for $d \leqslant 13$. Using a different approach, Pak and Soffer [72] recently provided a confirmation for $d \leqslant 16$. While Higman's conjecture remains open in general and despite some evidence suggesting that it may fail to hold for large d (see [72]), it nonetheless influenced and inspired numerous results on related questions; see, in particular, work of Isaacs [54, 55] on character degrees of so-called algebra groups and work of Goodwin and Röhrle [43–46] on conjugacy classes of unipotent elements in groups of Lie type.

Orbits of linear groups. All rings in this article are assumed to be commutative and unital. Let R be a ring and let V be an R-module with $|V| < \infty$. Given a linear group $G \leqslant \mathrm{GL}(V)$, it is a classical problem (for $R = \mathbf{F}_q$) to relate arithmetic properties of the number of orbits of G on its natural module V to geometric and group-theoretic properties of G; see e.g. [37] and the references therein. This problem is closely related to the enumeration of irreducible characters (and hence of conjugacy classes). In particular, if G is a finite p-group of nilpotency class less than p, then the Kirillov orbit method establishes a bijection between the ordinary irreducible characters of G and the coadjoint orbits of G on the dual of its associated Lie ring; cf. [39] and see [71] for applications of such techniques to the enumeration of characters and conjugacy classes.

Rank distributions and the average size of a kernel. In addition to group-theoretic problems such as those indicated above, this article is also inspired by questions and results from probabilistic linear algebra. For an elementary example, to the author's knowledge, the number

$$\prod_{i=0}^{r-1}(q^e - q^i)\frac{q^{d-i}-1}{q^{i+1}-1} \tag{1.1}$$

of $d \times e$ matrices of rank r with entries in a finite field $\mathbf{F}_q$ was first recorded by Landsberg [63]. More recently, probabilistic questions surrounding the distribution of ranks in sets of matrices over finite fields have been studied, see e.g. [3, 8, 18, 32] and [62, Ch. 3]; for applications, see [65, 85].

Let R be a ring, let V and W be R-modules with $|V|, |W| < \infty$, and let $M \subset \mathrm{Hom}(V, W)$ be a submodule. In the following, we are primarily interested in the case that R is finite and $M \subset \mathrm{M}_{d\times e}(R)$ acts by right-multiplication on $V = R^d$. The **a**verage **s**ize of the **k**ernel of the elements of M is

$$\mathrm{ask}(M) := \mathrm{ask}(M \mid V) := |M|^{-1} \sum_{a \in M} |\mathrm{Ker}(a)|.$$

Linial and Weitz [66] gave the following formula for $\mathrm{ask}(\mathrm{M}_{d\times e}(\mathbf{F}_q))$; the same result appeared (with a different proof) in a recent paper by Fulman and Goldstein [36, Lem. 3.2] which also contains further examples of $\mathrm{ask}(M)$.

Proposition 1.1. $\mathrm{ask}(\mathrm{M}_{d\times e}(\mathbf{F}_q)) = 1 + q^{d-e} - q^{-e}$.

As we will see later, for a linear p-group $G \leqslant \mathrm{GL}(V)$ with a sufficiently strong Lie theory, $|V/G|$ and $\mathrm{k}(G)$ are both instances of $\mathrm{ask}(\mathfrak{g})$ for suitable linear Lie algebras $\mathfrak{g}$.

Orbit-counting and conjugacy class zeta functions. In the literature, numbers of the form $|V/G|$, $\mathrm{k}(G)$, and $\mathrm{ask}(M \mid V)$ for R-modules V and W, a linear group $G \leqslant \mathrm{GL}(V)$, and $M \subset \mathrm{Hom}(V, W)$ were primarily studied in the case that $R = \mathbf{F}_q$ is a finite field. Instead of individual numbers, we consider families of such numbers obtained by replacing $\mathbf{F}_q$ by the finite quotients of suitable rings.

We will use the following notation throughout this article. Let K be a non-Archimedean local field and let $\mathfrak{O}$ be its valuation ring—equivalently, $\mathfrak{O}$ is a compact discrete valuation ring with field of fractions K; we occasionally write $\mathfrak{O}_K$ instead of $\mathfrak{O}$ and similarly below. For example, K could be the field $\mathbf{Q}_p$ of p-adic numbers (in which case $\mathfrak{O} = \mathbf{Z}_p$ is the ring of p-adic integers) or the field $\mathbf{F}_q(\!(z)\!)$ of formal Laurent series over $\mathbf{F}_q$ (in which case $\mathfrak{O} = \mathbf{F}_q[\![z]\!]$). Let $\mathfrak{P}$ denote the maximal ideal of $\mathfrak{O}$. Let $\mathfrak{K} := \mathfrak{O}/\mathfrak{P}$ be the residue field of K and let q and p denote the size and characteristic of $\mathfrak{K}$, respectively. We write $\mathfrak{P}^n = \mathfrak{P} \cdots \mathfrak{P}$ for the nth ideal power of $\mathfrak{P}$; on the other hand, $\mathfrak{O}^n = \mathfrak{O} \times \cdots \times \mathfrak{O}$ denotes the nth Cartesian power of $\mathfrak{O}$. Let $\mathfrak{O}_n = \mathfrak{O}/\mathfrak{P}^n$ and $\mathfrak{O}_\infty = \mathfrak{O}$.

Definition 1.2. Let $G \leqslant \mathrm{GL}_d(\mathfrak{O})$ be a subgroup.

(i) Let $G_n \leqslant \mathrm{GL}_d(\mathfrak{O}_n)$ denote the image of G under the natural map $\mathrm{GL}_d(\mathfrak{O}) \twoheadrightarrow \mathrm{GL}_d(\mathfrak{O}_n)$. The **conjugacy class zeta function** of G is $\mathsf{Z}^{\mathsf{cc}}_G(T) := \sum\limits_{n=0}^{\infty} \mathrm{k}(G_n) T^n$.

(ii) The **orbit-counting zeta function** of G is $\mathsf{Z}^{\mathsf{oc}}_G(T) := \sum\limits_{n=0}^{\infty} |\mathfrak{O}_n^d / G| T^n$.

Referring to these generating functions as "zeta functions" is justified by various properties recalled or established in the following (e.g. the existence of meromorphic continuation) for the associated Dirichlet series $\mathsf{Z}^{\mathsf{cc}}_{G}(q^{-s})$ and $\mathsf{Z}^{\mathsf{oc}}_{G}(q^{-s})$, at least in characteristic zero. Conjugacy class zeta functions were introduced by du Sautoy [27] who established their rationality for $\mathfrak{O} = \mathbf{Z}_p$. Berman et al. [6] investigated $\mathsf{Z}^{\mathsf{cc}}_{\mathsf{G}(\mathfrak{O})}(T)$ for Chevalley groups G. Lins [67, 68] recently determined $\mathsf{Z}^{\mathsf{cc}}_{\mathsf{G}(\mathfrak{O})}(T)$ for certain families of unipotent group schemes G. Special cases of the functions $\mathsf{Z}^{\mathsf{oc}}_{G}(T)$ have previously appeared in the literature. In particular, Avni et al. [2, Thms E, A.5] determined orbit-counting zeta functions associated with the coadjoint representation of GL_d and group schemes of the form GU_d for $d = 2, 3$.

Conjugacy class and orbit-counting zeta functions are natural analogues of the numbers of conjugacy classes and orbits of finite groups from above. For example, it is a natural generalisation of Higman's conjecture to ask, for each fixed d, whether $\mathsf{Z}^{\mathsf{cc}}_{\mathrm{U}_d(\mathfrak{O}_K)}(T)$ is given by a rational function in q_K and T as a function of K.

The definition of $\mathsf{Z}^{\mathsf{ask}}_{M}(T)$. We now introduce the protagonist of this article. Let V and W be finitely generated $\mathfrak{O}$-modules. We frequently write $V_n = V \otimes \mathfrak{O}_n$ and $W_n = W \otimes \mathfrak{O}_n$, where, in the absence of subscripts, tensor products are always taken over $\mathfrak{O}$. Given a submodule $M \subset \mathrm{Hom}(V, W)$, we let M_n denote the image of M under the natural map $\mathrm{Hom}(V, W) \to \mathrm{Hom}(V_n, W_n)$, $a \mapsto a \otimes \mathrm{id}_{\mathfrak{O}_n}$. Crucially, the module M_n does not merely depend on the abstract module M but rather on the given embedding of M into $\mathrm{Hom}(V, W)$. In particular, the natural surjection $M \otimes \mathfrak{O}_n \twoheadrightarrow M_n$ need not be an isomorphism; for example, if $M = \mathfrak{P} \subset \mathfrak{O} = \mathrm{End}(\mathfrak{O})$, then $M \otimes \mathfrak{O}_1$ is isomorphic to $\mathfrak{O}_1$ but $M_1 = 0$. In terms of matrices, for a submodule $M \subset \mathrm{M}_{d\times e}(\mathfrak{O})$, we obtain $M_n \subset \mathrm{M}_{d\times e}(\mathfrak{O}_n)$ by reducing the entries of all matrices in M modulo $\mathfrak{P}^n$. This article is devoted to generating functions of the following form.

Definition 1.3. Let $M \subset \mathrm{M}_{d\times e}(\mathfrak{O})$ be a submodule and $V = \mathfrak{O}^d$. Define the **ask zeta function** of M to be

$$\mathsf{Z}_M(T) := \mathsf{Z}^{\mathsf{ask}}_{M}(T) := \mathsf{Z}^{\mathsf{ask}}_{M \curvearrowright V}(T) := \sum_{n=0}^{\infty} \mathrm{ask}(M_n \mid V_n) T^n \in \mathbf{Q}[\![T]\!].$$

In contrast to the probabilistic flavour of the work on the numbers $\mathrm{ask}(M \mid V)$ cited above, our investigations of the functions $\mathsf{Z}^{\mathsf{ask}}_{M}(T)$ draw upon results and techniques that have been previously applied in asymptotic group theory and, specifically, the theory of zeta functions (representation zeta functions, in particular) of groups and other algebraic structures; for recent surveys of this area, see [61, 93, 94]. Conversely, our study of ask zeta functions contributes to asymptotic group theory: we will see that orbit-counting and conjugacy class zeta functions of suitable groups are instances of ask zeta functions.

Results I: fundamental properties and examples of ask zeta functions. Our central structural result on the functions $\mathsf{Z}^{\mathsf{ask}}_{M}(T)$ is the following.

Theorem 1.4. *Let $\mathfrak{O}$ be the valuation ring of a non-Archimedean local field of characteristic zero. Let $M \subset \mathrm{M}_{d\times e}(\mathfrak{O})$ be a submodule. Then $\mathsf{Z}^{\mathsf{ask}}_{M}(T)$ is rational, i.e. $\mathsf{Z}^{\mathsf{ask}}_{M}(T) \in \mathbf{Q}(T)$.*

For example, $\mathsf{Z}^{\mathsf{ask}}_{\{0_{d\times e}\}}(T) = 1/(1-q^d T)$. At the other extreme, we will obtain the following generalisation of Proposition 1.1.

Proposition 1.5. *Let $\mathfrak{O}$ be the valuation ring of a non-Archimedean local field of arbitrary characteristic. Let q be the residue field size of $\mathfrak{O}$. Then*

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{M}_{d\times e}(\mathfrak{O})}(T) = \frac{1-q^{-e}T}{(1-T)(1-q^{d-e}T)}. \tag{1.2}$$

Note that since $\mathsf{Z}^{\mathsf{ask}}_{\mathrm{M}_{d\times e}(\mathfrak{O})}(T) = 1 + (1+q^{d-e}-q^{-e})T + \mathcal{O}(T^2)$, Proposition 1.5 indeed generalises Proposition 1.1. Apart from proving Proposition 1.5, in §5, we will also determine $\mathsf{Z}^{\mathsf{ask}}_M(T)$ for traceless (Corollary 5.10), symmetric (Proposition 5.13), anti-symmetric (Proposition 5.11), upper triangular (Proposition 5.15), and diagonal (Corollary 5.17) matrices. We will also explain why many of our formulae are of the same shape as (1.2). Despite this wealth of explicit examples in arbitrary characteristic, the author does not know if Theorem 1.4 remains true in positive characteristic; see §4.3.4.

Our proofs of Theorem 1.4, Proposition 1.5, and various other results in this article rest upon expressing the functions $\mathsf{Z}^{\mathsf{ask}}_M(T)$ in terms of suitable integrals (Theorem 4.5). These integrals can then be studied using powerful techniques developed over the past decades, primarily in the context of Igusa's local zeta function (see [21, 52] for introductions). Our use of these techniques is similar to and inspired by their applications in the theory of zeta functions of groups and, in particular, the study of representation growth; see [1, 48, 57, 83, 92]. In particular, Theorem 1.4 follows from rationality results going back to Igusa and Denef. Using a theorem of Voll [92] we will furthermore see that the identity

$$\mathsf{Z}^{\mathsf{ask}}_{\mathrm{M}_{d\times e}(\mathfrak{O})}(T)\,\Bigg|_{(q,T)\to(q^{-1},T^{-1})} = (-q^d T)\cdot \mathsf{Z}^{\mathsf{ask}}_{\mathrm{M}_{d\times e}(\mathfrak{O})}(T) \tag{1.3}$$

is no coincidence (Theorem 4.18). Our p-adic formalism is also compatible with our previous computational work (summarised in [80]) which allows us to explicitly compute numerous further examples of $\mathsf{Z}^{\mathsf{ask}}_M(T)$; see §9 for some of these.

While "random matrices" over local fields have been studied before (see e.g. [34]), the author is not aware of previous applications of the particular techniques employed (and the point of view taken) here.

Results II: ask zeta functions and asymptotic group theory. We say that a formal power series $F(T) \in \mathbf{Q}[\![T]\!]$ has **bounded denominators** if there exists a non-zero $a \in \mathbf{Z}$ such that $aF(T) \in \mathbf{Z}[\![T]\!]$. As usual, for a ring R and R-module V, let $\mathfrak{gl}(V)$ denote the Lie algebra associated with the associative algebra $\mathrm{End}(V)$; that is, $\mathfrak{gl}(V) = \mathrm{End}(V)$ as R-modules and the Lie bracket of $\mathfrak{gl}(V)$ is defined in terms of multiplication in $\mathrm{End}(V)$ via $[a,b] = ab - ba$.

Theorem 1.6. *Let $\mathfrak{O}$ be the valuation ring of a non-Archimedean local field of characteristic zero. Let $\mathfrak{g} \subset \mathfrak{gl}_d(\mathfrak{O})$ be a Lie subalgebra. Then $\mathsf{Z}^{\mathsf{ask}}_{\mathfrak{g}}(T)$ has bounded denominators.*

Theorem 1.6 is based on a connection between $\mathsf{Z}^{\mathsf{ask}}_{\mathfrak{g}}(T)$ and orbit-counting zeta functions. For a sketch, let $G \leqslant \mathrm{GL}_d(\mathfrak{O})$ act on $V = \mathfrak{O}^d$. As before, we write $\mathfrak{O}_n = \mathfrak{O}/\mathfrak{P}^n$ and $V_n = V \otimes \mathfrak{O}_n$. Then G acts on each of the finite sets V_n and, extending our previous definition of the orbit-counting zeta function $\mathsf{Z}^{\mathsf{oc}}_G(T)$ (Definition 1.2(ii)), we let

$$\mathsf{Z}^{\mathsf{oc},m}_G(T) = \sum_{n=m}^{\infty} |V_n/G| \cdot T^{n-m} \in \mathbf{Z}[\![T]\!];$$

hence, $\mathsf{Z}^{\mathsf{oc}}_G(T) = \mathsf{Z}^{\mathsf{oc},0}_G(T)$. In the setting of Theorem 1.6, by linearising the orbit-counting lemma using p-adic Lie theory, we will see that for sufficiently large m, there exists $G^m \leqslant \mathrm{GL}_d(\mathfrak{O})$ with $q^{dm}\mathsf{Z}^{\mathsf{ask}}_{\mathfrak{g}}(T) = \mathsf{Z}^{\mathsf{oc},m}_{G^m}(T)$. Theorem 1.6 then follows immediately.

In addition to using group theory to deduce properties of ask zeta functions such as Theorem 1.6, we will see that, conversely, our methods for studying ask zeta functions allow us to deduce results on both orbit-counting and conjugacy class zeta functions. As we will now sketch, this direction is particularly fruitful for unipotent groups. For a Lie algebra $\mathfrak{g}$ over a ring R, let $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ denote its adjoint representation given by $\mathrm{ad}(a)\colon b \mapsto [b, a]$ for $a \in \mathfrak{g}$. Let $\mathfrak{n}_d(R) \subset \mathfrak{gl}_d(R)$ denote the Lie algebra of strictly upper triangular $d \times d$ matrices.

Theorem 1.7. *Let $\mathfrak{O}$ be the valuation ring of a local field of characteristic zero and residue characteristic p. Let $\mathfrak{g} \subset \mathfrak{n}_d(\mathfrak{O})$ be a Lie subalgebra and let $G := \exp(\mathfrak{g}) \leqslant \mathrm{U}_d(\mathfrak{O})$. Suppose that $p \geqslant d$ and that $\mathfrak{g}$ is an isolated submodule of $\mathfrak{n}_d(\mathfrak{O})$ (i.e. the $\mathfrak{O}$-module quotient $\mathfrak{n}_d(\mathfrak{O})/\mathfrak{g}$ is torsion-free). Then $\mathsf{Z}^{\mathsf{oc}}_G(T) = \mathsf{Z}^{\mathsf{ask}}_{\mathfrak{g}}(T)$ and $\mathsf{Z}^{\mathsf{cc}}_G(T) = \mathsf{Z}^{\mathsf{ask}}_{\mathrm{ad}(\mathfrak{g})}(T)$.*

We will apply Theorem 1.7 and the methods for computing $\mathsf{Z}^{\mathsf{ask}}_M(T)$ developed below in order to compute "generic" conjugacy class zeta functions arising from all unipotent algebraic groups of dimension at most 5 over a number field (see §9.3).

Due to the heavy reliance of the proofs of Theorems 1.6–1.7 on p-adic Lie theory, it is unclear to the author whether these results have analogues over local fields of positive characteristic.

Outline. In §§2–3, we collect elementary facts on $\mathrm{ask}(M \mid V)$ and $\mathsf{Z}^{\mathsf{ask}}_M(T)$. We then derive expressions for $\mathsf{Z}^{\mathsf{ask}}_M(T)$ in terms of suitable integrals in §4. In §5, we use these to compute explicit formulae for $\mathsf{Z}^{\mathsf{ask}}_M(T)$ for various modules M. Next, in §6, we discuss a relationship between the functions $\mathsf{Z}^{\mathsf{ask}}_M(T)$ and "constant rank spaces" studied extensively in the literature. A geometric source of interesting examples of ask zeta functions, determinantal hypersurfaces, is considered in §7. In §8, we explore the aforementioned connection between ask, conjugacy class, and orbit-counting zeta functions in characteristic zero. In particular, we prove Theorems 1.6–1.7. Finally, given that most of the explicit formulae for $\mathsf{Z}^{\mathsf{ask}}_M(T)$ obtained in §§5–6 are quite tame, §9 contains a number of more complicated examples of $\mathsf{Z}^{\mathsf{ask}}_M(T)$ and $\mathsf{Z}^{\mathsf{cc}}_G(T)$.

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in an earlier version, and to the referee for numerous insightful suggestions.

2 Elementary properties of average sizes of kernels

We collect some elementary observations on average sizes of kernels. Throughout, unless otherwise stated, let R be a ring, let V and W be R-modules with $|V|, |W| < \infty$, and let $M \subset \mathrm{Hom}(V, W)$ be a submodule.

2.1 Rank varieties

In the case of a finite field $R = \mathbf{F}_q$, $\mathrm{ask}(M \mid V)$ admits a natural geometric interpretation. Namely, by choosing a basis of M, we may identify $M = \mathbf{A}^{\ell}_{\mathbf{F}_q}(\mathbf{F}_q)$, where $\ell = \dim_{\mathbf{F}_q}(M)$ and $\mathbf{A}^{\ell}_{\mathbf{F}_q} = \mathrm{Spec}(\mathbf{F}_q[X_1, \dots, X_\ell])$. We may then decompose $\mathbf{A}^{\ell}_{\mathbf{F}_q} = \coprod_{i=0}^{r} V_i$, where V_i is the subvariety of maps of rank i. (Note that if $M = \mathrm{M}_{d\times e}(\mathbf{F}_q)$, then $\#V_r(\mathbf{F}_q)$ is given by (1.1).) Then $\mathrm{ask}(M \mid V) = \sum_{i=0}^{d} \#V_i(\mathbf{F}_q) \cdot q^{d-i-\ell}$; in fact, by replacing q by q^f on the right-hand side, we express $\mathrm{ask}(M \otimes_{\mathbf{F}_q} \mathbf{F}_{q^f} \mid V \otimes_{\mathbf{F}_q} \mathbf{F}_{q^f})$ using a formula which is valid for any $f \geqslant 1$. However, even for $M = \mathrm{M}_{d\times e}(\mathbf{F}_q)$, this approach yields a fairly complicated interpretation of Proposition 1.1.

2.2 Kernels and orbits

One simple yet crucial observation contained in the proof of [66, Thm 1.1] is the following connection between the sizes of the kernels $\mathrm{Ker}(a)$ $(a \in M)$ and those of the "orbits" $\boldsymbol{x}M := \{\boldsymbol{x}a : a \in M\} \approx M/\,\mathrm{c}_M(\boldsymbol{x})$, where $\boldsymbol{x} \in V$ and $\mathrm{c}_M(\boldsymbol{x}) := \{a \in M : \boldsymbol{x}a = 0\}$; note that in contrast to orbits under group actions, the sets $\boldsymbol{x}M$ always overlap.

Lemma 2.1 (Cf. [66, Thm 1.1])**.** $\mathrm{ask}(M \mid V) = \sum\limits_{\boldsymbol{x}\in V} |\boldsymbol{x}M|^{-1}$.

We give two proofs of this lemma. The first is a combinatorial version of a probabilistic argument in the proof of [66, Thm 1.1]. We include it here since our terminology is different from theirs; similar arguments appear in [69].

First proof of Lemma 2.1. By computing $\#\{(\boldsymbol{x}, a) \in V \times M : \boldsymbol{x}a = 0\}$ in two ways, we obtain $\sum\limits_{a\in M} |\mathrm{Ker}(a)| = \sum\limits_{\boldsymbol{x}\in V} |\mathrm{c}_M(\boldsymbol{x})|$. Since $\boldsymbol{x}M \approx M/\,\mathrm{c}_M(\boldsymbol{x})$ as R-modules, we have $|\mathrm{c}_M(\boldsymbol{x})| = |M|/|\boldsymbol{x}M|$ and thus $\mathrm{ask}(M \mid V) = |M|^{-1} \sum\limits_{\boldsymbol{x}\in V} |\mathrm{c}_M(\boldsymbol{x})| = \sum\limits_{\boldsymbol{x}\in V} |\boldsymbol{x}M|^{-1}$. ♦

Our second proof of Lemma 2.1 already hints at the connection between average sizes of kernels and orbits of linear groups, a subject further explored in §8. Recall that for a finite group G acting on a finite set X, the orbit-counting lemma asserts that $|X/G| = |G|^{-1} \sum\limits_{g\in G} |\mathrm{Fix}(g)|$, where $\mathrm{Fix}(g) = \{x \in X : xg = x\}$.

Second proof of Lemma 2.1. The rule $a \mapsto a^* := \left[\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right]$ yields an isomorphism of $(M, +)$ onto a subgroup M^* of $\mathrm{GL}(V \oplus W)$. We claim that the natural bijection $V \oplus W \to \coprod\limits_{V} W$

induces a bijection $(V \oplus W)/M^* \to \coprod_{\boldsymbol{x}\in V} W/\boldsymbol{x}M$. Indeed, $(\boldsymbol{x},\boldsymbol{y})a^* = (\boldsymbol{x},\boldsymbol{y}+\boldsymbol{x}a)$ for $(\boldsymbol{x},\boldsymbol{y}) \in V \oplus W$. As $\mathrm{Fix}(a^*) = \mathrm{Ker}(a) \oplus W$, the orbit-counting lemma yields

$$|W| \sum_{\boldsymbol{x}\in V} |\boldsymbol{x}M|^{-1} = |(V\oplus W)/M^*| = |M^*|^{-1} \sum_{a\in M} |\mathrm{Fix}(a^*)| = |W| \cdot \mathrm{ask}(M \mid V). \qquad \blacklozenge$$

In order to deduce Proposition 1.1 from Lemma 2.1, note that $\boldsymbol{x}\,\mathrm{M}_{d\times e}(\mathbf{F}_q) = \mathbf{F}_q^e$ for each non-zero $\boldsymbol{x} \in \mathbf{F}_q^d$ whence $\mathrm{ask}(\mathrm{M}_{d\times e}(\mathbf{F}_q)) = 1 + (q^d-1)q^{-e}$.

2.3 Interlude: rank distributions and hyperoctahedral groups

We discuss combinatorial consequences of Proposition 1.1.

Reminder: the hyperoctahedral groups B_n. For background and details on the following, see [11, §3] or [7, §8.1]. The **hyperoctahedral group** $\mathrm{B}_n = \{\pm 1\} \wr \mathrm{S}_n$ is the group of signed permutations on n letters; we regard B_n as a subgroup of the symmetric group on $\{\pm 1, \dots, \pm n\}$ and as a Coxeter group in the usual way (see [7, p. 246]). For $\sigma \in \mathrm{B}_n$, we write $\sigma = [1^\sigma, \dots, n^\sigma]$. For $\sigma \in \mathrm{B}_n$, let $\mathrm{len}(\sigma)$ denote the (Coxeter) **length** of σ (see [11, Prop. 3.1] for a combinatorial description), let $\mathrm{N}(\sigma) := \#\{i \in \{1,\dots,n\} : i^\sigma < 0\}$, and let $\mathrm{Des}(\sigma) := \{i \in \{0,\dots,n-1\} : i^\sigma > (i+1)^\sigma\}$ (where we wrote $0^\sigma = 0$) denote the **descent set** of σ. For $I \subset \{0,\dots,n\}$, define the **quotient** $\mathrm{B}_n^{I^c} := \{\sigma \in \mathrm{B}_n : \mathrm{Des}(\sigma) \subset I\}$ (see [7, §2.4]). The identity $1 \in \mathrm{B}_n$ is the unique element of length 0. Moreover, since $\mathrm{Des}(1) = \varnothing$, the identity is contained in each set $\mathrm{B}_n^{I^c}$.

Let $\mathrm{M}_d^{\mathrm{rk}=i}(\mathbf{F}_q) := \{a \in \mathrm{M}_d(\mathbf{F}_q) : \mathrm{rk}(a) = i\}$. As explained in [13, §3.2], for $i = 0,\dots,d$,

$$\left|\mathrm{M}_d^{\mathrm{rk}=d-i}(\mathbf{F}_q)\right| = q^{d^2-i^2} \sum_{\sigma\in \mathrm{B}_d^{\{i\}^c}} (-1)^{\mathrm{N}(\sigma)} q^{-\mathrm{len}(\sigma)} \tag{2.1}$$

whence

$$\begin{aligned} \mathrm{ask}(\mathrm{M}_d(\mathbf{F}_q)) &= q^{-d^2} \sum_{i=0}^{d} \left|\mathrm{M}_d^{\mathrm{rk}=d-i}(\mathbf{F}_q)\right| q^i \\ &= \sum_{i=0}^{d} q^{i-i^2} \sum_{\sigma\in \mathrm{B}_d^{\{i\}^c}} (-1)^{\mathrm{N}(\sigma)} q^{-\mathrm{len}(\sigma)}. \end{aligned} \tag{2.2}$$

On the other hand, by Proposition 1.1, $\mathrm{ask}(\mathrm{M}_d(\mathbf{F}_q)) = 2 - q^{-d}$. The right-hand side of (2.2) is a polynomial in $\mathbf{Z}[q^{-1}]$, the constant term, 2, of which arises from $\sigma = 1 \in \mathrm{B}_n^{\{i\}^c}$ and $i = 0, 1$. However, the fact that the other terms of (2.2) add up to $-q^{-d}$ seems much less transparent. Consider, for example, the case $d = 2$. For $i = 0, 1$, we have $q^{i-i^2} = 1$ and the contributions to the right-hand side of (2.2) are exactly the terms $(-1)^{\mathrm{N}(\sigma)} q^{-\mathrm{len}(\sigma)}$ indicated in the following tables.

$\sigma \in \mathrm{B}_2^{\{0\}^c}$	$(-1)^{\mathrm{N}(\sigma)}q^{-\operatorname{len}(\sigma)}$
1	$+1$
$[-1,2]$	$-q^{-1}$
$[-2,1]$	$-q^{-2}$
$[-2,-1]$	$+q^{-3}$

$\sigma \in \mathrm{B}_2^{\{1\}^c}$	$(-1)^{\mathrm{N}(\sigma)}q^{-\operatorname{len}(\sigma)}$
1	$+1$
$[2,1]$	$+q^{-1}$
$[2,-1]$	$-q^{-2}$
$[1,-2]$	$-q^{-3}$

For $i = 2$, $\mathrm{B}_2^{\{2\}^c} = \{1\}$ and the contribution to the right-hand side of (2.2) is a single summand q^{-2}. While we can therefore confirm that

$$2 - q^{-2} = \operatorname{ask}(\mathrm{M}_2(\mathbf{F}_q)) = (1 - q^{-1} - q^{-2} + q^{-3}) + (1 + q^{-1} - q^{-2} - q^{-3}) + q^{-2},$$

the author is unable to provide a combinatorial explanation of this numerical coincidence.

A further source of such examples is given by analogues of (2.1) for the numbers of traceless, antisymmetric, and symmetric $d \times d$ matrices over $\mathbf{F}_q$, respectively, due to Carnevale et al. [13, §3.2]; cf. [83]. The average sizes of the kernels of all these spaces of matrices are known or can be deduced as by-products or our investigations in §5.3 below.

2.4 Direct sums

Lemma 2.2. *Let V' and W' be R-modules with $|V'|, |W'| < \infty$. Let $M' \subset \operatorname{Hom}(V', W')$ be a submodule. We regard $M \oplus M'$ as a submodule of $\operatorname{Hom}(V \oplus V', W \oplus W')$ in the natural way. Then $\operatorname{ask}(M \oplus M' \mid V \oplus V') = \operatorname{ask}(M \mid V) \cdot \operatorname{ask}(M' \mid V')$.*

Proof.

$$\begin{aligned}
\operatorname{ask}(M \oplus M' \mid V \oplus V') &= |M \oplus M'|^{-1} \cdot \sum_{(a,a') \in M \oplus M'} |\operatorname{Ker}(a \oplus a')| \\
&= |M|^{-1}|M'|^{-1} \cdot \sum_{(a,a') \in M \oplus M'} |\operatorname{Ker}(a)| \cdot |\operatorname{Ker}(a')| \\
&= \operatorname{ask}(M \mid V) \cdot \operatorname{ask}(M' \mid V'). \qquad \blacklozenge
\end{aligned}$$

Corollary 2.3. *Let R be finite, $M \subset \mathrm{M}_{d \times e}(R)$ be a submodule, and $\tilde{M} \subset \mathrm{M}_{(d+1) \times e}(R)$ be obtained from M by adding a zero row to the elements of M in some fixed position. Then $\operatorname{ask}(\tilde{M}) = \operatorname{ask}(M) \cdot |R|$.* ♦

2.5 Matrix transposes

Following Kaplansky [58], we call R an **elementary divisor ring** if for each $a \in \mathrm{M}_{d \times e}(R)$ (and all $d, e \geqslant 1$), there exist $u \in \mathrm{GL}_d(R)$ and $v \in \mathrm{GL}_e(R)$ such that uav is a diagonal matrix (padded with zeros according to the shape of a). For example, any quotient of a principal ideal domain is an elementary divisor ring (regardless of whether the quotient is an integral domain or not). Write $a^\top$ for the transpose of a.

Lemma 2.4. *Let R be a finite elementary divisor ring and let $M \subset \mathrm{M}_{d \times e}(R)$ be a submodule. Write $V = R^d$ and $W = R^e$. Then $\operatorname{ask}(M^\top \mid W) = \operatorname{ask}(M \mid V) \cdot |R|^{e-d}$.*

Proof. Let $a = \begin{bmatrix} \operatorname{diag}(a_1,\dots,a_r) & 0 \\ 0 & 0 \end{bmatrix} \in \mathrm{M}_{d\times e}(R)$. Then $\operatorname{Ker}(a)$ consists of those $\boldsymbol{x} \in V$ with $a_i x_i = 0$ for $1 \leqslant i \leqslant r$ and $\operatorname{Ker}(a^\top)$ consists of those $\boldsymbol{y} \in W$ with $a_i y_i = 0$ for $1 \leqslant i \leqslant r$. ♦

2.6 Reduction modulo $\mathfrak{a}$ and base change $R \to R/\mathfrak{a}$

Let V and W be finitely generated R-modules, the underlying sets of which need not be finite. As before, let $M \subset \operatorname{Hom}(V, W)$ be a submodule. Let $\mathfrak{a} \triangleleft R$ with $|R/\mathfrak{a}| < \infty$. Define $V_\mathfrak{a} := V \otimes_R R/\mathfrak{a}$, $W_\mathfrak{a} := W \otimes_R R/\mathfrak{a}$, and let $M_\mathfrak{a}$ be the image of the natural map $M \hookrightarrow \operatorname{Hom}(V, W) \to \operatorname{Hom}(V_\mathfrak{a}, W_\mathfrak{a})$. In general, the natural surjection $M \otimes_R R/\mathfrak{a} \twoheadrightarrow M_\mathfrak{a}$ need not be injective (see the example on p. 4). However, if M is finitely generated, then $M \otimes_R R/\mathfrak{a}$ is finite and we obtain the following expression for $\operatorname{ask}(M_\mathfrak{a} \mid V_\mathfrak{a})$.

Lemma 2.5. *Suppose that M is finitely generated. Then*

$$\operatorname{ask}(M_\mathfrak{a} \mid V_\mathfrak{a}) = |M \otimes_R R/\mathfrak{a}|^{-1} \sum_{\bar{a} \in M \otimes_R R/\mathfrak{a}} |\operatorname{Ker}(\bar{a} \mid V_\mathfrak{a})|.$$ ♦

3 Basic algebraic and analytic properties of $\mathsf{Z}_M(T)$ and $\zeta_M(s)$

3.1 Average sizes of kernels and Dirichlet series: $\zeta_M(s)$

While our main focus is on the generating functions $\mathsf{Z}_M(T)$ from the introduction, it is natural to also consider a global analogue. First suppose that R is a ring which contains only finitely many ideals $\mathfrak{a}$ of a given finite norm $|R/\mathfrak{a}|$. Given a submodule $M \subset \mathrm{M}_{d\times e}(R)$ acting on $V = R^d$ and an ideal $\mathfrak{a} \triangleleft R$, let $V_\mathfrak{a} = (R/\mathfrak{a})^d$, $W_\mathfrak{a} = (R/\mathfrak{a})^e$, and let $M_\mathfrak{a}$ denote the image of the natural map $M \hookrightarrow \mathrm{M}_{d\times e}(R) \twoheadrightarrow \mathrm{M}_{d\times e}(R/\mathfrak{a})$ (cf. §2.6).

Definition 3.1.

(i) Define a formal Dirichlet series

$$\zeta_M(s) = \sum_{\mathfrak{a}} \operatorname{ask}(M_\mathfrak{a} \mid V_\mathfrak{a}) \cdot |R/\mathfrak{a}|^{-s},$$

where the sum extends over the ideals of finite norm of R and s denotes a complex variable.

(ii) Let $\alpha_M \in [-\infty, \infty]$ denote the abscissa of convergence of $\zeta_M(s)$.

3.2 Abscissae of convergence: local case

Let K be a local field of arbitrary characteristic with valuation ring $\mathfrak{O}$ and residue field size q. Let $M \subset \mathrm{M}_{d\times e}(\mathfrak{O})$ be a submodule acting on $V = \mathfrak{O}^d$. Then $\zeta_M(s) = \mathsf{Z}_M(q^{-s})$. Moreover, if $\mathfrak{O}$ has characteristic zero, then Theorem 1.4 (proved in §4.3.3) implies that α_M is precisely the largest real pole of (the meromorphic continuation of) $\zeta_M(s)$.

Recall that, unless otherwise indicated, tensors products are taken over $\mathfrak{O}$.

Definition 3.2. The **generic orbit rank** of M is $\operatorname{gor}(M) := \max_{\boldsymbol{x}\in V} \dim_K(\boldsymbol{x}M \otimes K)$.

Our choice of terminology will be justified by Proposition 4.13.

Proposition 3.3. $\max(d - \operatorname{gor}(M), 0) \leqslant \alpha_M \leqslant d$.

Proof. The upper bound follows since $\operatorname{ask}(M_n \mid V_n) \leqslant |V_n| = q^{nd}$ and $\sum_{n=0}^{\infty} q^{n(d-s)}$ converges for $\operatorname{Re}(s) > d$. Similarly, the lower bound follows from Lemma 2.1 and $\operatorname{ask}(M_n \mid V_n) \geqslant \max(|V_n|/q^{n\operatorname{gor}(M)}, 1)$. ♦

Let $0 \leqslant r \leqslant e$. Let $M \subset \mathrm{M}_{d\times e}(\mathfrak{O})$ be obtained from $\mathrm{M}_{d\times r}(\mathfrak{O})$ by inserting $e - r$ zero columns in some fixed positions. Then $\operatorname{gor}(M) = r$, $\zeta_M(s) = \zeta_{\mathrm{M}_{d\times r}(\mathfrak{O})}(s)$, and it will follow from Proposition 1.5 that $\alpha_M = \max(d - r, 0)$. In particular, the bounds in Proposition 3.3 are optimal. We note that Example 9.1 below illustrates that the meromorphic continuation of $\zeta_M(s)$ (cf. Theorem 1.4) may have real poles less than $d - e$.

3.3 Abscissae of convergence in the global case and Euler products

Let k be a number field with ring of integers $\mathfrak{o}$. Let $\mathcal{V}_k$ denote the set of non-Archimedean places of k. For $v \in \mathcal{V}_k$, let k_v be the v-adic completion of k and let $\mathfrak{o}_v$ be its valuation ring. We let q_v denote the size of the residue field $\mathfrak{K}_v$ of k_v. For an $\mathfrak{o}$-module U and $v \in \mathcal{V}_k$, we write $U_v := U \otimes_{\mathfrak{o}} \mathfrak{o}_v$ (regarded as an $\mathfrak{o}_v$-module).

Let $M \subset \mathrm{M}_{d\times e}(\mathfrak{o})$ be a submodule. For $v \in \mathcal{V}_k$, we may identify M_v with the $\mathfrak{o}_v$-submodule of $\mathrm{M}_{d\times e}(\mathfrak{o}_v)$ generated by M.

Proposition 3.4. *Let $M \subset \mathrm{M}_{d\times e}(\mathfrak{o})$ be a submodule. Then:*

(i) $\alpha_M \leqslant d + 1$.

(ii) $\zeta_M(s) = \prod_{v\in\mathcal{V}_k} \zeta_{M_v}(s)$.

Proof. Let $V = \mathfrak{o}^d$. The proof of (i) is similar to that of Proposition 3.3. Namely, for each $\mathfrak{a} \triangleleft \mathfrak{o}$, $\operatorname{ask}(M_{\mathfrak{a}} \mid V_{\mathfrak{a}}) \leqslant |\mathfrak{o}/\mathfrak{a}|^d$ and $\sum_{\mathfrak{a}} |\mathfrak{o}/\mathfrak{a}|^{d-s} = \zeta_k(s-d)$ converges for $\operatorname{Re}(s) > d+1$, where $\zeta_k(s)$ is the Dedekind zeta function of k. For (ii), it suffices to show that for non-zero coprime ideals $\mathfrak{a}, \mathfrak{b} \triangleleft \mathfrak{o}$, $\operatorname{ask}(M_{\mathfrak{ab}} \mid V_{\mathfrak{ab}}) = \operatorname{ask}(M_{\mathfrak{a}} \mid V_{\mathfrak{a}}) \cdot \operatorname{ask}(M_{\mathfrak{b}} \mid V_{\mathfrak{b}})$.

To that end, the natural isomorphism $\mathfrak{o}/\mathfrak{ab} \to \mathfrak{o}/\mathfrak{a} \times \mathfrak{o}/\mathfrak{b}$ yields an ($\mathfrak{o}$-module) isomorphism $M \otimes_{\mathfrak{o}} \mathfrak{o}/\mathfrak{ab} \to (M \otimes_{\mathfrak{o}} \mathfrak{o}/\mathfrak{a}) \times (M \otimes_{\mathfrak{o}} \mathfrak{o}/\mathfrak{b})$ which is compatible with the corresponding isomorphism $V_{\mathfrak{ab}} \to V_{\mathfrak{a}} \times V_{\mathfrak{b}}$ in the evident way. Hence, for $\bar{a} \in M \otimes_{\mathfrak{o}} \mathfrak{o}/\mathfrak{ab}$ corresponding to $(\bar{a}_{\mathfrak{a}}, \bar{a}_{\mathfrak{b}}) \in (M \otimes_{\mathfrak{o}} \mathfrak{o}/\mathfrak{a}) \times (M \otimes_{\mathfrak{o}} \mathfrak{o}/\mathfrak{b})$, we obtain an isomorphism $\operatorname{Ker}(\bar{a} \mid V_{\mathfrak{ab}}) \to \operatorname{Ker}(\bar{a}_{\mathfrak{a}} \mid V_{\mathfrak{a}}) \times \operatorname{Ker}(\bar{a}_{\mathfrak{b}} \mid V_{\mathfrak{b}})$. Part (ii) thus follows from Lemma 2.5. ♦

Example 3.5. Let $\zeta_k(s)$ denote the Dedekind zeta function of k. Then Proposition 1.5 and Proposition 3.4(ii) imply that $\zeta_{\mathrm{M}_{d\times e}(\mathfrak{o})}(s) = \zeta_k(s)\zeta_k(s-d+e)/\zeta_k(s+e)$.

Further analytic properties of $\zeta_M(s)$ in a global setting will be derived in §4.5.

3.4 Hadamard products

Recall that the **Hadamard product** $F(T) \star G(T)$ of formal power series $F(T) = \sum_{n=0}^{\infty} a_n T^n$ and $G(T) = \sum_{n=0}^{\infty} b_n T^n$ with coefficients in some common ring is

$$F(T) \star G(T) = \sum_{n=0}^{\infty} a_n b_n T^n.$$

The following is an immediate consequence of Lemma 2.2.

Corollary 3.6. *Let K be a local field of arbitrary characteristic with valuation ring $\mathfrak{O}$. Let $M = A \oplus B \subset \mathrm{M}_{d+e}(\mathfrak{O})$ for submodules $A \subset \mathrm{M}_d(\mathfrak{O})$ and $B \subset \mathrm{M}_e(\mathfrak{O})$. Then $\mathsf{Z}_M(T) = \mathsf{Z}_A(T) \star \mathsf{Z}_B(T)$.* ♦

We note that Hadamard products of rational generating functions are well-known to be rational (see [82, Prop. 4.2.5]).

Corollary 3.7. *Let $M \subset \mathrm{M}_{d\times e}(\mathfrak{O})$ be a submodule. Define $f = \max(d, e)$. Let $\tilde{M} \subset \mathrm{M}_f(\mathfrak{O})$ be obtained from M by adding $f - d$ zero rows and $f - e$ zero columns to the elements of M in some fixed positions. Then $\mathsf{Z}_{\tilde{M}}(T) = \mathsf{Z}_M(q^{f-d}T)$.* ♦

Thus, various questions on the series $\mathsf{Z}_M(T)$ are reduced to the case of square matrices.

3.5 Rescaling

Let $\mathfrak{O}$ be the valuation ring of a non-Archimedean local field K of arbitrary characteristic. Let $M \subset \mathrm{M}_{d\times e}(\mathfrak{O})$ be a submodule, $V = \mathfrak{O}^d$, and $W = \mathfrak{O}^e$.

Definition 3.8. For $m \geqslant 0$, let $\mathsf{Z}_M^m(T) := \sum\limits_{n=m}^{\infty} \mathrm{ask}(M_n \mid V_n) \cdot T^{n-m} \in \mathbf{Q}[\![T]\!]$.

Note that $\mathsf{Z}_M(T) = \mathsf{Z}_M^0(T)$. For an $\mathfrak{O}$-module U, let $U^m = \mathfrak{P}^m U$ and write U_n^m for the common value of $(U^m)_n$ and $(U_n)^m$. Clearly, if $n \leqslant m$, then $\mathrm{ask}(M_n^m \mid V_n) = \mathrm{ask}(\{0\} \mid V_n) = |V_n| = q^{nd}$.

Proposition 3.9. $\mathsf{Z}_{M^m}^m(T) = q^{dm} \cdot \mathsf{Z}_M(T)$.

Proof. It suffices to show that $\mathrm{ask}(M_n^m \mid V_n) = q^{dm} \cdot \mathrm{ask}(M_{n-m} \mid V_{n-m})$ for $n \geqslant m$. Choose $\pi \in \mathfrak{P} \setminus \mathfrak{P}^2$. Observe that multiplication by π^m induces an $\mathfrak{O}$-module isomorphism $M_{n-m} \to M_n^m$ and a monomorphism $V_{n-m} \to V_n$ with image V_n^m. For $a \in M$,

$$\begin{aligned}\mathrm{Ker}(\pi^m a \mid V_n) &= \{\boldsymbol{x} \in V_n : \boldsymbol{x}(\pi^m a) \equiv 0 \pmod{W^n}\}\\ &= \{\boldsymbol{x} \in V_n : \boldsymbol{x}a \equiv 0 \pmod{W^{n-m}}\}\\ &= \{\boldsymbol{x} \in V_n : \boldsymbol{x} + V^{n-m} \in \mathrm{Ker}(a \mid V_{n-m})\}\end{aligned}$$

has size $|\mathrm{Ker}(a \mid V_{n-m})| \cdot q^{dm}$. We conclude that

$$\begin{aligned}\mathrm{ask}(M_n^m \mid V_n) &= |M_n^m|^{-1} \sum_{a \in M_n^m} |\mathrm{Ker}(a \mid V_n)|\\ &= |M_{n-m}|^{-1} \sum_{a \in M_{n-m}} |\mathrm{Ker}(a \mid V_{n-m})| \cdot q^{dm}\\ &= q^{dm} \cdot \mathrm{ask}(M_{n-m} \mid V_{n-m}).\end{aligned}$$ ♦

4 Rationality of $\mathsf{Z}_M(T)$ and p-adic integration

Unless otherwise stated, in this section, K is a non-Archimedean local field of arbitrary characteristic with valuation ring $\mathfrak{O}$. Given a submodule $M \subset \mathrm{M}_{d\times e}(\mathfrak{O})$, we use the original definition of $\mathrm{ask}(M \mid V)$ as well as the alternative formula in Lemma 2.1 to derive two types of expressions for $\mathsf{Z}_M(T)$ in terms of p-adic integrals.

In §4.1, we describe a general setting for rewriting certain generating functions as integrals. By specialising to the case at hand, we obtain, in §4.2, two expressions for $\mathsf{Z}_M(T)$ (Theorem 4.5) in terms of functions K_M and O_M that we introduce. In §4.3, we derive explicit formulae (in terms of the absolute value of K and polynomials over $\mathfrak{O}$) for these functions. These formulae serve two purposes. First, using established rationality results from p-adic integration, they allow us to deduce Theorem 1.4. Secondly, these formulae, in particular the one based on O_M (and hence on Lemma 2.1), lie at the heart of explicit formulae such as Proposition 1.5 in §5.

4.1 Generating functions and p-adic integrals

Let Z be a free $\mathfrak{O}$-module of finite rank d and let $U \subset Z$ be a submodule. By the elementary divisor theorem, there exists a unique sequence $(\lambda_1, \dots, \lambda_d)$ with $0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_d \leqslant \infty$ such that $Z/U \approx \bigoplus_{i=1}^d \mathfrak{O}_{\lambda_i}$ as $\mathfrak{O}$-modules; recall that $\mathfrak{O}_\infty = \mathfrak{O}$. We call $(\lambda_1, \dots, \lambda_d)$ the **submodule type** of U within Z. Recall that the **isolator** $\mathrm{iso}_Z(U)$ of U in Z is the preimage of the torsion submodule of Z/U under the natural map $Z \twoheadrightarrow Z/U$. Equivalently, $\mathrm{iso}_Z(U)$ is the smallest direct summand of Z which contains U. Recall that U is **isolated** in Z if and only if $U = \mathrm{iso}_Z(U)$; this is equivalent to each λ_i from above being either 0 or ∞. If U is isolated in Z, then we may naturally identify $U \otimes \mathfrak{O}_n$ with the image U_n of $U \hookrightarrow Z \twoheadrightarrow Z_n := Z \otimes \mathfrak{O}_n$; to see that this identification may fail if U is not isolated, consider e.g. $U = \mathfrak{P} \subset \mathfrak{O} = Z$ and $n = 1$. If U is isolated in Z and U has rank ℓ, say, then $|U_n| = q^{\ell n}$. As we will now see, the general case is only slightly more complicated.

We let ν denote the valuation on K with value group $\mathbf{Z}$. Let $|\cdot|$ be the absolute value on K with $|\pi| = q^{-1}$ for $\pi \in \mathfrak{P} \setminus \mathfrak{P}^2$; we write $\|A\| = \sup(|a| : a \in A)$.

Lemma 4.1. *Let $U \subset Z$ be a submodule. Suppose that U has submodule type $(\lambda_1, \dots, \lambda_\ell)$ within $\mathrm{iso}_Z(U)$. Let U_n denote the image of the natural map $U \hookrightarrow Z \twoheadrightarrow Z_n := Z \otimes \mathfrak{O}_n$. Then $|U_n| = q^{\sum\limits_{i=1}^{\ell}(n-\min(\lambda_i,n))}$. In particular, for $y \in \mathfrak{O}$ with $\nu(y) = n$, $|U_n| = \prod\limits_{i=1}^{\ell} \frac{\|\pi^{\lambda_i}, y\|}{|y|}$.*

Proof. Clearly, $|U_n| = q^{\sum_{i=1}^{\ell}\min(n-\lambda_i,0)}$ and the first identity follows from $\min(0, n-a) = n - \min(n, a)$; the second claim then follows immediately. ⧫

The following result is concerned with generating functions associated with a given family of "weight functions" $f_n \colon U_n \to \mathbf{R}_{\geqslant 0}$. Given a free $\mathfrak{O}$-module W of finite rank, let μ_W denote the Haar measure on W with $\mu_W(W) = 1$.

Lemma 4.2. *Let $U \subset Z$ be a submodule. Suppose that U has submodule type $(\lambda_1, \dots, \lambda_\ell)$ within $\mathrm{iso}_Z(U)$. Let U_n denote the image of $U \hookrightarrow Z \twoheadrightarrow Z_n := Z \otimes \mathfrak{O}_n$ and let $\pi_n \colon U \to U_n$ denote the natural map. Let $N \geqslant 0$ and, for $n \geqslant N$, let $f_n \colon U_n \to \mathbf{R}_{\geqslant 0}$ be given. Define*

$$F \colon U \times \mathfrak{P}^N \setminus \{0\} \to \mathbf{R}_{\geqslant 0}, \quad (\boldsymbol{x}, y) \mapsto f_{\nu(y)}(\pi_{\nu(y)}(\boldsymbol{x}))$$

and extend F to a map $U \times \mathfrak{P}^N \to \mathbf{R}_{\geqslant 0}$ via $F(\boldsymbol{x}, 0) \equiv 0$. Let

$$g \colon \mathfrak{P}^N \times \mathbf{C} \to \mathbf{R}_{\geqslant 0}, \quad (y, s) \mapsto |y|^{s-\ell-1} \cdot \prod_{i=1}^{\ell} \|\pi^{\lambda_i}, y\|,$$

where we set $g(0, s) \equiv 0$. (Note that $g(y, s) = |y|^{s-1} \cdot |U_{\nu(y)}|$ for $y \neq 0$ by Lemma 4.1.)

Suppose that $\delta \geqslant 0$ satisfies $\sum\limits_{\bar{\boldsymbol{x}} \in U_n} f_n(\bar{\boldsymbol{x}}) = \mathcal{O}(q^{\delta n})$. Then, writing $t = q^{-s}$, for all $s \in \mathbf{C}$ with $\mathrm{Re}(s) > \delta$,

$$\sum_{n=N}^{\infty} \sum_{\bar{\boldsymbol{x}} \in U_n} f_n(\bar{\boldsymbol{x}}) t^n = (1 - q^{-1})^{-1} \int\limits_{U \times \mathfrak{P}^N} F(\boldsymbol{x}, y) \cdot g(y, s) \,\mathrm{d}\mu_{U \times \mathfrak{O}}(\boldsymbol{x}, y). \tag{4.1}$$

Proof. First note that the left-hand side of (4.1) converges for $\mathrm{Re}(s) > \delta$. Further note that we may ignore the case $y = 0$ on the right-hand side as it occurs on a set of measure zero.

Let $U^n = \mathrm{Ker}(\pi_n \colon U \twoheadrightarrow U_n)$. Given $(\boldsymbol{x}, y) \in U \times \mathfrak{P}^N \setminus \{0\}$ with $n := \nu(y)$, the map F is constant on the open set $(\boldsymbol{x} + U^n) \times (y + \mathfrak{P}^{n+1})$; in particular, F is measurable.

Let $\mathcal{R}_n \subset U$ be a complete and irredundant set of representatives for the cosets of U^n and let $\mathcal{W}_n = \mathfrak{P}^n \cdot \mathfrak{O}^\times = \mathfrak{P}^n \setminus \mathfrak{P}^{n+1}$. By Lemma 4.1, $\mu_U(U^n) = |U_n|^{-1} = \prod\limits_{i=1}^{\ell} \frac{|y|}{\|\pi^{\lambda_i}, y\|}$ for any $y \in \mathcal{W}_n$; moreover, $\mu_{\mathfrak{O}}(\mathcal{W}_n) = (1 - q^{-1}) q^{-n}$. The claim now follows via

$$\begin{aligned}
\int\limits_{U \times \mathfrak{P}^N} F(\boldsymbol{x}, y) \cdot g(y, s) \,\mathrm{d}\mu_{U \times \mathfrak{O}}(\boldsymbol{x}, y) &= \sum_{n=N}^{\infty} \sum_{\boldsymbol{x} \in \mathcal{R}_n} \int\limits_{(\boldsymbol{x} + U^n) \times \mathcal{W}_n} f_n(\pi_n(\boldsymbol{x})) \cdot \underbrace{g(y, s)}_{= |U_n| \cdot (qt)^n} \,\mathrm{d}\mu_{U \times \mathfrak{O}}(\boldsymbol{x}, y) \\
&= (1 - q^{-1}) \sum_{n=N}^{\infty} \sum_{\bar{\boldsymbol{x}} \in U_n} f_n(\bar{\boldsymbol{x}}) \cdot \mu(U^n \times \mathcal{W}_n) \cdot |U_n| \cdot (qt)^n. \\
&= (1 - q^{-1}) \sum_{n=N}^{\infty} \sum_{\bar{\boldsymbol{x}} \in U_n} f_n(\bar{\boldsymbol{x}}) t^n. \qquad \blacklozenge
\end{aligned}$$

Remark 4.3. The introduction of the additional variable y to express a generating function as an integral in Lemma 4.2 mimics similar formulae of Jaikin-Zapirain [57, §4] and Voll [92, §2.2].

4.2 $\mathsf{Z}_M(T)$ and the functions K_M and O_M

Let $M \subset \mathrm{M}_{d \times e}(\mathfrak{O})$ be a submodule, $V = \mathfrak{O}^d$, and $W = \mathfrak{O}^e$.

Definition 4.4. Define

$$\mathrm{K}_M\colon M\times\mathfrak{O}\to\mathbf{N}_0\cup\{\infty\},\quad (a,y)\mapsto \#\mathrm{Ker}\Big(V\otimes\mathfrak{O}/(y)\xrightarrow{a\otimes\mathfrak{O}/(y)} W\otimes\mathfrak{O}/(y)\Big)\text{ and}$$

$$\mathrm{O}_M\colon V\times\mathfrak{O}\to\mathbf{N}_0\cup\{\infty\},\quad (\boldsymbol{x},y)\mapsto \#\,\mathrm{Im}\Big(\boldsymbol{x}M\hookrightarrow W\twoheadrightarrow W\otimes\mathfrak{O}/(y)\Big).$$

Note that for $y\neq 0$, $1\leqslant \mathrm{K}_M(a,y)\leqslant |y|^{-d}$ and $1\leqslant \mathrm{O}_M(\boldsymbol{x},y)\leqslant |y|^{-\operatorname{gor}(M)}$ (see Definition 3.2); these are the same estimates as in Proposition 3.3.

Using Lemmas 2.1 and 4.2, we obtain the following formulae for $\mathsf{Z}_M(t)$ (where $t=q^{-s}$).

Theorem 4.5. *For $s\in\mathbf{C}$ with $\mathrm{Re}(s)>d$,*

$$(1-q^{-1})\cdot\mathsf{Z}_M(q^{-s})=\int\limits_{M\times\mathfrak{O}}|y|^{s-1}\cdot\mathrm{K}_M(a,y)\,\mathrm{d}\mu_{M\times\mathfrak{O}}(a,y)=\int\limits_{V\times\mathfrak{O}}\frac{|y|^{s-d-1}}{\mathrm{O}_M(\boldsymbol{x},y)}\,\mathrm{d}\mu_{V\times\mathfrak{O}}(\boldsymbol{x},y).$$

Proof. The given formulae for $(1-q^{-1})\cdot\mathsf{Z}_M(q^{-s})$ are based on $\operatorname{ask}(M_n\mid V_n)=|M_n|^{-1}\cdot\sum_{a\in M_n}|\mathrm{Ker}(a)|$ and $\operatorname{ask}(M_n\mid V_n)=\sum_{\bar{\boldsymbol{x}}\in V_n}|\bar{\boldsymbol{x}}M_n|^{-1}$ (Lemma 2.1), respectively. In detail, the first equality follows from Lemma 4.2 with $U=M$, $Z=\mathrm{M}_{d\times e}(\mathfrak{O})$, $F(a,y)=|M_{\nu(y)}|^{-1}\,\mathrm{K}_M(a,y)$ and $g(y,s)=|y|^{s-1}\cdot|M_{\nu(y)}|$ (for $y\neq 0$). For the second equality, use Lemma 4.2 with $U=Z=V$, $F(\boldsymbol{x},y)=|(\boldsymbol{x}+y\mathfrak{O}^d)M_{\nu(y)}|^{-1}=\mathrm{O}_M(\boldsymbol{x},y)^{-1}$ and $g(y,s)=|y|^{s-d-1}$ (for $y\neq 0$). ♦

4.3 Explicit formulae for K_M and O_M

As before, let $M\subset\mathrm{M}_{d\times e}(\mathfrak{O})$ be a submodule. In order to use Theorem 4.5 for theoretical investigations or explicit computations of $\mathsf{Z}_M(T)$, we need to produce sufficiently explicit formulae for $\mathrm{K}_M(a,y)$ or $\mathrm{O}_M(\boldsymbol{x},y)$.

4.3.1 The sizes of kernels and images

Let $a\in\mathrm{M}_{d\times e}(\mathfrak{O})$ have rank r over K. Fix $\pi\in\mathfrak{P}\setminus\mathfrak{P}^2$. By the elementary divisor theorem, there are $0\leqslant\lambda_1\leqslant\cdots\leqslant\lambda_r$, $u\in\mathrm{GL}_d(\mathfrak{O})$, and $v\in\mathrm{GL}_e(\mathfrak{O})$ such that

$$uav=\begin{bmatrix}\operatorname{diag}(\pi^{\lambda_1},\dots,\pi^{\lambda_r}) & 0\\ 0 & 0\end{bmatrix}. \tag{4.2}$$

We call $(\lambda_1,\dots,\lambda_r)$ the **equivalence type** of a.

For a set of polynomials $\boldsymbol{f}(\boldsymbol{Y})$, we write $\boldsymbol{f}(\boldsymbol{y})=\{f(\boldsymbol{y}):f\in\boldsymbol{f}\}$.

Lemma 4.6. *Let $a\in\mathrm{M}_{d\times e}(\mathfrak{O})$ have rank r over K and equivalence type $(\lambda_1,\dots,\lambda_r)$. Let $\boldsymbol{f}_i(\boldsymbol{Y})\subset\mathbf{Z}[\boldsymbol{Y}]:=\mathbf{Z}[Y_{ij}:1\leqslant i\leqslant d,\,1\leqslant j\leqslant e]$ be the set of non-zero $i\times i$ minors of the generic $d\times e$ matrix $[Y_{ij}]\in\mathrm{M}_{d\times e}(\mathbf{Z}[\boldsymbol{Y}])$. (Hence, $\boldsymbol{f}_0(\boldsymbol{Y})=\{1\}$.) Let $a_n\in\mathrm{M}_{d\times e}(\mathfrak{O}_n)$ be the image of the matrix a under the natural map $\mathrm{M}_{d\times e}(\mathfrak{O})\twoheadrightarrow\mathrm{M}_{d\times e}(\mathfrak{O}_n)$. Then:*

(i) $\|\boldsymbol{f}_i(a)\|=q^{-\lambda_1-\cdots-\lambda_i}$ for $0\leqslant i\leqslant r$.

(ii) (Cf. [92, §2.2]) $\frac{\|\boldsymbol{f}_{i-1}(a)\|}{\|\boldsymbol{f}_i(a)\cup\pi^n\boldsymbol{f}_{i-1}(a)\|} = q^{\min(\lambda_i,n)}$ *for* $1 \leqslant i \leqslant r$.

(iii) $|\mathrm{Ker}(a_n)| = q^{\min(\lambda_1,n)+\cdots+\min(\lambda_r,n)+(d-r)n}$.

(iv) $|\mathrm{Im}(a_n)| = q^{rn-(\min(\lambda_1,n)+\cdots+\min(\lambda_r,n))}$.

Proof. The first part is elementary linear algebra. Part (ii) then follows from

$$q^{-\lambda_1-\cdots-\lambda_{i-1}+\min(\lambda_1+\cdots+\lambda_i,\lambda_1+\cdots+\lambda_{i-1}+n)} = q^{\min(\lambda_i,n)}.$$

For $0 \leqslant i \leqslant n$, the map $\mathfrak{O}_n \to \mathfrak{O}_n$ given by multiplication by π^i has kernel and image size q^i and q^{n-i}, respectively. Parts (iii)–(iv) thus follow from equation (4.2). ♦

4.3.2 A formula for K_M

We use Lemma 4.6 in order to derive a formula for $\mathrm{K}_M(a,y)$.

Definition 4.7. The **generic element rank** of M is $\mathrm{grk}(M) := \max\limits_{a\in M}\mathrm{rk}_K(a)$.

By the following, $\mathrm{grk}(M)$ is the rank of a "generic" matrix in M in any meaningful sense.

Proposition 4.8. *Let* $(a_1,\dots,a_\ell)$ *be an* $\mathfrak{O}$*-basis of* M *and let* $\lambda_1,\dots,\lambda_\ell$ *be algebraically independent over* K*. Then:*

(i) $\mathrm{grk}(M) = \mathrm{rk}_{K(\lambda_1,\dots,\lambda_\ell)}(\lambda_1 a_1+\cdots+\lambda_\ell a_\ell)$.

(ii) $\mu_M(\{a\in M : \mathrm{rk}_K(a) < \mathrm{grk}(M)\}) = 0$.

Proof. Let r denote the right-hand side in (i). Then r is the largest number such that some $r\times r$ minor, $m(\lambda_1,\dots,\lambda_\ell)$ say, of $\lambda_1a_1+\cdots+\lambda_\ell a_\ell$ is non-zero. In particular, $\mathrm{grk}(M)\leqslant r$. Conversely, since $m(\lambda_1,\dots,\lambda_\ell)\neq 0$, we find $c_1,\dots,c_\ell\in\mathfrak{O}$ with $m(c_1,\dots,c_\ell)\neq 0$ whence $\mathrm{grk}(M)\geqslant r$. Finally, the well-known fact (provable using induction and Fubini's theorem) that the zero locus of a non-zero polynomial over K has measure zero implies (ii). ♦

Excluding a set of measure zero, we thus obtain the following formula for $\mathrm{K}_M(a,y)$.

Corollary 4.9. *Let* $N = \{a\in M : \mathrm{rk}_K(a) < \mathrm{grk}(M)\}$ *and let* $\boldsymbol{f}_i(\boldsymbol{Y})$ *be the set of non-zero* $i\times i$ *minors of the generic* $d\times e$ *matrix. Then for all* $a\in M\setminus N$ *and* $y\in\mathfrak{O}\setminus\{0\}$,

$$\mathrm{K}_M(a,y) = |y|^{\mathrm{grk}(M)-d}\cdot\prod_{i=1}^{\mathrm{grk}(M)}\frac{\|\boldsymbol{f}_{i-1}(a)\|}{\|\boldsymbol{f}_i(a)\cup y\boldsymbol{f}_{i-1}(a)\|}.$$

Proof. Immediate from Lemma 4.6(ii)–(iii). ♦

Hence, using Theorem 4.5, we conclude that

$$\mathsf{Z}_M(q^{-s}) = (1-q^{-1})^{-1}\int\limits_{M\times\mathfrak{O}}|y|^{s+\mathrm{grk}(M)-d-1}\cdot\prod_{i=1}^{\mathrm{grk}(M)}\frac{\|\boldsymbol{f}_{i-1}(a)\|}{\|\boldsymbol{f}_i(a)\cup y\boldsymbol{f}_{i-1}(a)\|}\,\mathrm{d}\mu_{M\times\mathfrak{O}}(a,y). \quad (4.3)$$

4.3.3 Rationality and variation of the place

As in the proof of Proposition 4.8, we may replace the $\boldsymbol{f}_i(\boldsymbol{Y})$ in (4.3) by polynomials in a chosen system of coordinates of M. We may thus interpret the integral in (4.3) as being defined in terms of valuations of polynomial expressions in $\dim_K(M \otimes K) + 1$ variables. Integrals of this form have been studied extensively. In particular, using well-known results going back to work of Igusa and Denef (see [21]), initially for a single polynomial and later extended to families of polynomials (see, in particular, work of du Sautoy and Grunewald [28], Veys and Zúñiga-Galindo [88], and Voll [92]), we obtain the following two results, the first of which implies and refines Theorem 1.4.

Theorem 4.10. *Let $\mathfrak{O}$ be the valuation ring of a local field of characteristic zero and let $M \subset \mathrm{M}_{d\times e}(\mathfrak{O})$ be a submodule. Then $\mathsf{Z}_M(T) \in \mathbf{Q}(T)$. More precisely, there exist $f(T) \in \mathbf{Z}[T]$, non-zero $(a_1, b_1), \dots, (a_r, b_r)$ with $(a_i, b_i) \in \mathbf{Z} \times \mathbf{N}_0$, and $m \in \mathbf{N}_0$ such that*

$$\mathsf{Z}_M(T) = \frac{f(T)}{q^m(1 - q^{a_1}T^{b_1}) \cdots (1 - q^{a_r}T^{b_r})}. \qquad \blacklozenge$$

Moreover, in a global setting, the dependence of Euler factors on the place is as follows.

Theorem 4.11. *Let k be a number field with ring of integers $\mathfrak{o}$. Recall the notation from §3.3. Let $M \subset \mathrm{M}_{d\times e}(\mathfrak{o})$ be a submodule. There are separated $\mathfrak{o}$-schemes $V_1, \dots, V_r$ of finite type and rational functions $W_1(X,T), \dots, W_r(X,T) \in \mathbf{Q}(X,T)$ (which can be written over denominators of the same shape as those in Theorem 4.10) such that the following holds: for almost all $v \in \mathcal{V}_k$,*

$$\mathsf{Z}_{M_v}(T) = \sum_{i=1}^{r} \#V_i(\mathfrak{K}_v) \cdot W_i(q_v, T). \tag{4.4}$$

$\blacklozenge$

4.3.4 Local fields of positive characteristic

Suppose that $M \subset \mathrm{M}_{d\times e}(\mathfrak{O})$ is a submodule as before but that $K = \mathfrak{K}[\![z]\!]$ is a local field of *positive* characteristic. Then the techniques cited above to establish rationality in Theorem 4.10 do not apply to $\mathsf{Z}_M(T)$ and indeed, the author does not know if $\mathsf{Z}_M(T)$ is necessarily rational in positive characteristic. By combining Igusa's original proof of the rationality of his local zeta function (see [21, §1.3] for a modern account) and ideas of du Sautoy and Grunewald [28, §2], we obtain the following sufficient condition for rationality of ask zeta functions over $\mathfrak{O}$: if every hypersurface embedded inside some affine space over K admits an embedded resolution of singularities over K, then $\mathsf{Z}_M(T)$ is rational for all modules $M \subset \mathrm{M}_{d\times e}(\mathfrak{O})$. The status of resolution of singularities in positive characteristic is presently unresolved; see e.g. [49].

In contrast, to such unresolved issues, ask zeta functions in "large" positive characteristic arising from global models in characteristic zero are amenable to existing techniques. Namely, by applying powerful model-theoretic transfer principles such as [15, Thm 9.2.4] to our integrals, we obtain the following.

Theorem 4.12. *Let k be a number field with ring of integers $\mathfrak{o}$. Recall the notation from §3.3. Let $M \subset \mathrm{M}_{d\times e}(\mathfrak{o})$ be a submodule. Then for almost all $v \in \mathcal{V}_k$, $\mathsf{Z}_{M_v}(T) = \mathsf{Z}_{M\otimes_{\mathfrak{o}}\mathfrak{K}_v[\![z]\!]}(T)$. In particular, $\mathsf{Z}_{M\otimes_{\mathfrak{o}}\mathfrak{K}_v[\![z]\!]}(T)$ is rational for almost all $v \in \mathcal{V}_k$.* ♦

We note that Theorems 4.11–4.12 both behave well under local base extensions; cf. [77, Thm 2.3] and [95, Rem. 1.6].

4.3.5 A formula for O_M

As in the case of K_M, we can produce a formula for O_M.

Let $\boldsymbol{X} = (X_1, \dots, X_d)$ be algebraically independent over K. Let $C(\boldsymbol{X}) \in \mathrm{M}_{\ell\times e}(\mathfrak{O}[\boldsymbol{X}])$ with $\mathfrak{O}[\boldsymbol{X}]^d C(\boldsymbol{X}) = \boldsymbol{X}\cdot(M \otimes \mathfrak{O}[\boldsymbol{X}])$. For example, we may choose generators $a_1, \dots, a_\ell$ of M as an $\mathfrak{O}$-module and take

$$C(\boldsymbol{X}) = \begin{bmatrix} \boldsymbol{X}a_1 \\ \vdots \\ \boldsymbol{X}a_\ell \end{bmatrix} \in \mathrm{M}_{\ell\times e}(\mathfrak{O}[\boldsymbol{X}]).$$

Let $\boldsymbol{g}_i(\boldsymbol{X})$ be the set of non-zero $i\times i$ minors of $C(\boldsymbol{X})$. Note that if $\boldsymbol{x} \in V$, then $\boldsymbol{x}M$ is the row span of $C(\boldsymbol{x})$ over $\mathfrak{O}$ so that, in particular, $\dim_K(\boldsymbol{x}M \otimes K) = \mathrm{rk}_K(C(\boldsymbol{x}))$.

Recall the definition of $\mathrm{gor}(M)$ in Definition 3.2. The following is proved in the same way as Proposition 4.8.

Proposition 4.13. *Let $\boldsymbol{X} = (X_1, \dots, X_d)$ be algebraically independent over K. Then:*

(i) $\mathrm{gor}(M) = \dim_{K(\boldsymbol{X})}\big(\boldsymbol{X}\cdot(M\otimes K(\boldsymbol{X}))\big) = \mathrm{rk}_{K(\boldsymbol{X})}(C(\boldsymbol{X}))$.

(ii) $\mu_V(\{\boldsymbol{x} \in V : \dim_K(\boldsymbol{x}M \otimes K) < \mathrm{gor}(M)\}) = 0$. ♦

The following analogue of Corollary 4.9 is obtained using Lemma 4.6(ii),(iv).

Corollary 4.14. *Let $Z = \{\boldsymbol{x} \in V : \dim_K(\boldsymbol{x}M\otimes K) < \mathrm{gor}(M)\}$. Then for all $\boldsymbol{x} \in V\setminus Z$ and $y \in \mathfrak{O}\setminus\{0\}$,*

$$\mathrm{O}_M(\boldsymbol{x}, y) = |y|^{-\mathrm{gor}(M)} \prod_{i=1}^{\mathrm{gor}(M)} \frac{\|\boldsymbol{g}_i(\boldsymbol{x}) \cup y\boldsymbol{g}_{i-1}(\boldsymbol{x})\|}{\|\boldsymbol{g}_{i-1}(\boldsymbol{x})\|}. \tag{4.5}$$

♦

Theorem 4.5 thus provides us with the following counterpart of (4.3):

$$\mathsf{Z}_M(q^{-s}) = (1-q^{-1})^{-1}\int_{V\times\mathfrak{O}} |y|^{s+\mathrm{gor}(M)-d-1} \cdot \prod_{i=1}^{\mathrm{gor}(M)} \frac{\|\boldsymbol{g}_{i-1}(\boldsymbol{x})\|}{\|\boldsymbol{g}_i(\boldsymbol{x}) \cup y\boldsymbol{g}_{i-1}(\boldsymbol{x})\|}\, \mathrm{d}\mu_{V\times\mathfrak{O}}(\boldsymbol{x}, y). \tag{4.6}$$

Despite the essentially identical shapes of the integrals in (4.3) and (4.6), either type might be vastly more useful for explicit computations of specific examples. In particular, §5 is concerned with examples of $\mathsf{Z}_M(T)$ that can be easily computed using (4.6) and §6 considers the analogous situations for (4.3). A very similar phenomenon was exploited by O'Brien and Voll [71, §5] in their enumeration of conjugacy classes of certain relatively free p-groups.

Remark 4.15. We note that the integrals in (4.3)–(4.6) are almost of the same shape as those in [1, Eqn (1.4)]. These similarities can be clarified further by rewriting our integrals slightly; see the proof of Theorem 4.18 below. We further that the role of the matrix $C(\boldsymbol{X})$ here is similar to that of $A(\mathbf{X})$ in [71, Def. 2.1].

4.4 Projectivisation

For later applications, in the following, we record "projective" versions of the integrals in Theorem 4.5 in the spirit of Voll's formalism [92, §2.2], as rewritten by Avni et al. [1, §3.2].

Let $M \subset \mathrm{M}_{d\times e}(\mathfrak{O})$ be a submodule and $V = \mathfrak{O}^d$. The following lemma is elementary.

Lemma 4.16. *Let $\boldsymbol{x} \in V$, $a \in M$, $y \in \mathfrak{O} \setminus \{0\}$, and $\pi \in \mathfrak{P} \setminus \mathfrak{P}^2$. Then:*

(i) $\mathrm{O}_M(\pi\boldsymbol{x}, \pi y) = \mathrm{O}_M(\boldsymbol{x}, y)$.

(ii) $\mathrm{K}_M(\pi a, \pi y) = q^d \cdot \mathrm{K}_M(a, y)$.

Proof. Let $W := \mathfrak{O}^e$.

(i) For a submodule $U \subset W$ and $z \in \mathfrak{O}$, let U_z denote the image of U in $W \otimes \mathfrak{O}/(z)$. Since multiplication by π induces an isomorphism $U_z \approx (\pi U)_{\pi z}$, the claim follows by taking $U = \boldsymbol{x}M$ and $z = y$.

(ii) Let $\boldsymbol{x} \in V$. Then $\boldsymbol{x}(\pi a) \equiv 0 \pmod{\pi y W}$ if and only if $\boldsymbol{x}a \equiv 0 \pmod{yW}$. Clearly, each residue class modulo yW of such elements has precisely q^d lifts modulo $\pi y W$. ♦

Proposition 4.17. *Let $\ell := \dim_K(M \otimes K)$. Then:*

(i) $$(1 - q^{-s}) \cdot \mathsf{Z}_M(q^{-s}) = 1 + (1 - q^{-1})^{-1} \int\limits_{(V\setminus\mathfrak{P}V)\times\mathfrak{P}} \frac{|y|^{s-d-1}}{\mathrm{O}_M(\boldsymbol{x}, y)} \, \mathrm{d}\mu_{V\times\mathfrak{O}}(\boldsymbol{x}, y).$$

(ii) $$(1 - q^{d-\ell-s}) \cdot \mathsf{Z}_M(q^{-s}) = 1 + (1 - q^{-1})^{-1} \int\limits_{(M\setminus\mathfrak{P}M)\times\mathfrak{P}} |y|^{s-1} \, \mathrm{K}_M(a, y) \, \mathrm{d}\mu_{M\times\mathfrak{O}}(a, y).$$

Proof. Write $t := q^{-s}$.

(i) First, $\mathrm{O}_M(\boldsymbol{x}, y) = 1$ for $\boldsymbol{x} \in V$ and $y \in \mathfrak{O}^\times$. In the following, we write o as a shorthand for $\frac{|y|^{s-d-1}}{\mathrm{O}_M(\boldsymbol{x},y)} \, \mathrm{d}\mu_{V\times\mathfrak{O}}(\boldsymbol{x}, y)$. Using Lemma 4.16 and a change of variables, we find that $\int\limits_{\mathfrak{P}V\times\mathfrak{P}} o = t \cdot \int\limits_{V\times\mathfrak{O}} o$. The claim then follows from Theorem 4.5 and

$$\begin{aligned} \int\limits_{V\times\mathfrak{O}} o &= \int\limits_{V\times\mathfrak{O}^\times} o + \int\limits_{(V\setminus\mathfrak{P}V)\times\mathfrak{P}} o + \int\limits_{\mathfrak{P}V\times\mathfrak{P}} o \\ &= (1 - q^{-1}) + \int\limits_{(V\setminus\mathfrak{P}V)\times\mathfrak{P}} o + t \cdot \int\limits_{V\times\mathfrak{O}} o. \end{aligned}$$

(ii) For $a \in M$ and $y \in \mathfrak{O}^\times$, $\mathrm{K}_M(a,y) = 1$. One may then proceed similarly to (i) using

$$\int\limits_{\mathfrak{P}M \times \mathfrak{P}} \kappa = q^{d-\ell} t \cdot \int\limits_{M \times \mathfrak{O}} \kappa,$$

where $\kappa = |y|^{s-1}\, \mathrm{K}_M(a,y)\, \mathrm{d}\mu_{M\times\mathfrak{O}}(a,y)$. ♦

4.5 Local functional equations and global analytic properties

Functional equations under "inversion of the prime" are a common (but not universal) phenomenon in the theory of local zeta functions. Denef and Meuser [23] showed that for a homogeneous polynomial over a number field, almost all of its associated local Igusa zeta functions satisfy such a functional equation. Vastly generalising their result, Voll [92] established functional equations for numerous types of zeta functions arising in asymptotic algebra and expressible in terms of p-adic integrals. For further positive results establishing such functional equations, see, in particular, work of du Sautoy and Lubotzky [30], Voll [95], Avni et al. [1, §4], and Stasinski and Voll [83, Thm A]. Using the formalism developed above, we may deduce the following; recall the notation from §3.3.

Theorem 4.18. *Let k be a number field with ring of integers $\mathfrak{o}$. Let $M \subset \mathrm{M}_{d\times e}(\mathfrak{o})$ be a submodule. Then for almost all $v \in \mathcal{V}_k$,*

$$\mathsf{Z}_{M_v}(T)\Big|_{(q_v,T)\to(q_v^{-1},T^{-1})} = (-q_v^d T)\cdot \mathsf{Z}_{M_v}(T).$$

Remark 4.19.

(i) The operation "$q_v \to q_v^{-1}$" can be unambiguously defined in terms of an arbitrary formula of the form (4.4); see [23,92] and cf. [77, Cor. 4.3]. If $\mathsf{Z}_{M_v}(T) = W(q_v,T)$ for almost all $v \in \mathcal{V}_k$ and some $W(X,T) \in \mathbf{Q}(X,T)$, then Theorem 4.18 asserts that $W(X^{-1},T^{-1}) = (-X^d T)\cdot W(X,T)$; see [77, §4] and cf. (1.3).

(ii) Using Theorem 4.11–4.12, Theorem 4.18 also establishes, for almost all $v \in \mathcal{V}_k$, a functional equation for $\mathsf{Z}_{M\otimes_{\mathfrak{o}}\mathfrak{K}_v[\![z]\!]}(T)$; cf. [95, Cor. 1.3].

Proof of Theorem 4.18. We use Voll's results from [92, §2.1]. Let $K = k_v$ for $v \in \mathcal{V}_k$ and let $\mathfrak{O}$ be as before. Let $H_v(s)$ denote the right-hand side in Proposition 4.17(i). Using the surjection $\mathrm{GL}_d(\mathfrak{O}) \to V \setminus \mathfrak{P}V$ which sends a matrix to its first column, we rewrite $H_v(s)$ in terms of an integral over $\mathrm{GL}_d(\mathfrak{O})\times\mathfrak{P}$; cf. [1, §4]. In the setting of the explicit formula for $\mathrm{O}_{M_v}(\boldsymbol{x},y)$ derived in §4.3.5, we may assume that $C(\boldsymbol{X})$ is a matrix of linear forms whence each $\boldsymbol{g}_i(\boldsymbol{X})$ consists of homogeneous polynomials of degree i. This allows us to use [92, Cor. 2.4] which shows that $H_v(s)\big|_{q_v\to q_v^{-1}} = q^d H_v(s)$ for almost all $v \in \mathcal{V}_k$. ♦

Based on work of du Sautoy and Grunewald [28], Duong and Voll [33] studied analytic properties of Euler products of functions of the same form as the right-hand sides in Proposition 4.17(i)–(ii). In particular, their findings allow us to deduce the following.

Theorem 4.20 (Cf. [33, Thm A]). *Let k be a number field with ring of integers $\mathfrak{o}$. Let $M \subset \mathrm{M}_{d\times e}(\mathfrak{o})$ be a submodule and $V = \mathfrak{o}^d$.*

(i) The abscissa of convergence α_M of $\zeta_M(s)$ is a rational number.

(ii) There exists $\delta > 0$ such that $\zeta_M(s)$ admits meromorphic continuation to $\{s \in \mathbf{C} : \mathrm{Re}(s) > \alpha_M - \delta\}$. This continued function has a pole of order β_M, say, at $s = \alpha_M$ but no other poles on the line $\mathrm{Re}(s) = \alpha_M$.

(iii) α_M and β_M are (and δ can be chosen to be) invariant under base change of M from $\mathfrak{o}$ to the ring of integers of an arbitrary finite extension of k.

4.6 Reduced and topological ask zeta functions

Let k be a number field with ring of integers $\mathfrak{o}$. Recall the notation from §3.3. Given a suitable family of local zeta functions indexed by places $v \in \mathcal{V}_k$, associated "reduced" and "topological" zeta functions are obtained by passing to two different limits "$q_v \to 1$". The original topological zeta functions of Denef and Loeser [22] are singularity invariants attached to polynomials. Later, du Sautoy and Loeser [29] defined topological subobject zeta functions of algebraic structures. Reduced subobject zeta functions were introduced by Evseev [35]. Topological and reduced representation zeta functions of unipotent groups were studied by the author in [75] and [79, §7], respectively.

The techniques from p-adic integration used above are similar to those employed in the study of representation zeta functions of unipotent groups. As a consequence, we immediately obtain adequate notions of reduced and topological ask zeta functions which we now briefly discuss. Let $M \subset \mathrm{M}_{d\times e}(\mathfrak{o})$ be a submodule.

Topological ask zeta functions. Informally, the topological ask zeta function $\zeta_M^{\mathrm{top}}(s) \in \mathbf{Q}(s)$ of M is the constant term of $(1 - q_v^{-1})\zeta_{M_v}(s)$ as a series in $q_v - 1$; for a rigorous definition, combine the formalism developed in [74, §5] (and summarised in [80, §4.2]), Proposition 4.17, and [75, Pf of Lem. 3.4]. For example, Proposition 1.5 implies that

$$\zeta^{\mathrm{top}}_{\mathrm{M}_{d\times e}(\mathbf{Z})}(s) = \frac{s+e}{s(s-d+e)}.$$

We note that, as in the case of subobject [74, Prop. 5.19] and representation zeta functions [75, Prop. 4.3], the topological ask zeta function of M only depends on $M \otimes_{\mathfrak{o}} \bar{k}$, where $\bar{k}$ is an algebraic closure of k.

Reduced ask zeta functions. Informally, the reduced ask zeta function $\mathsf{Z}_M^{\mathrm{red}}(T) \in \mathbf{Q}(T)$ is obtained from the formal power series $\mathsf{Z}_{M_v}(T)$ by applying a limit "$q_v \to 1$" to each coefficient. In the present context, this process can be formalised just as in the case of representation zeta functions of unipotent groups (see [79, §7]). Moreover, Proposition 4.17 and a variation of [79, Pf of Thm 7.3] (which relies heavily on arguments due to Duong and Voll [33]) show that in fact $\mathsf{Z}_M^{\mathrm{red}}(T) = 1/(1-T)$ for any M. This is intuitively plausible: if $\tilde{M} \subset \mathrm{M}_{d\times e}(\mathfrak{O}_n)$ is a submodule, then the group $(\mathfrak{O}/\mathfrak{P})^\times$ acts freely on

$\tilde{M} \setminus \{0\}$ and preserves kernels whence $|\tilde{M}| \cdot \mathrm{ask}(\tilde{M}) \equiv |\tilde{V}| \pmod{(q-1)}$, where $\tilde{V} = \mathfrak{O}_n^d$. In particular, one would expect any reasonable limit of $\mathrm{ask}(\tilde{M})$ as "$q \to 1$" to be 1.

5 Full matrix algebras, classical Lie algebras, and relatives

In this section, let $\mathfrak{O}$ be the valuation ring of a non-Archimedean local field of arbitrary characteristic. Apart from proving Proposition 1.5, we compute examples of $\mathsf{Z}_{\mathfrak{g}}(T)$, where $\mathfrak{g}$ ranges over various infinite families of matrix Lie algebras. At the heart of these computations lies the notion of "O-maximality" introduced in §5.1.

5.1 O-maximality

Let $M \subset \mathrm{M}_{d\times e}(\mathfrak{O})$ be a submodule and $V = \mathfrak{O}^d$. As we will now see, $\mathrm{O}_M(\boldsymbol{x}, y)$ is (generically) as large as possible if and only if $\mathsf{Z}_M(T)$ coincides with $\mathsf{Z}_{\mathrm{M}_{d\times \mathrm{gor}(M)}}(T)$.

Lemma 5.1. *Let $\boldsymbol{x} \in V$ and $y \in \mathfrak{O} \setminus \{0\}$. Then $\mathrm{O}_M(\boldsymbol{x}, y) \leqslant |y|^{-\mathrm{gor}(M)} \|\boldsymbol{x}, y\|^{\mathrm{gor}(M)}$.*

Proof. Let $C(\boldsymbol{X}) \in \mathrm{M}_{\ell\times e}(\mathfrak{O}[\boldsymbol{X}])$ with $\mathfrak{O}[\boldsymbol{X}]^d C(\boldsymbol{X}) = \boldsymbol{X} \cdot (M \otimes \mathfrak{O}[\boldsymbol{X}])$. We may assume that $C(\boldsymbol{x}) \neq 0$ since otherwise $\mathrm{O}_M(\boldsymbol{x}, y) = 1$. As $0_{\mathfrak{O}^d} \cdot M = \{0_{\mathfrak{O}^e}\}$, the constant terms of all non-zero polynomials in $C(\boldsymbol{X})$ vanish whence $\|C(\boldsymbol{x})\| \leqslant \|\boldsymbol{x}\|$. Thus, if $C(\boldsymbol{x})$ has equivalence type $(\lambda_1, \dots, \lambda_r)$, then $|\pi^{\lambda_i}| \leqslant |\pi^{\lambda_1}| = \|C(\boldsymbol{x})\| \leqslant \|\boldsymbol{x}\|$ for $1 \leqslant i \leqslant r$. Define m and n via $q^{-m} = \|\boldsymbol{x}\|$ and $n = \nu(y)$. Then, by Lemma 4.6(iv),

$$\mathrm{O}_M(\boldsymbol{x}, y) = q^{rn - \sum_{i=1}^{r} \min(\lambda_i, n)} \leqslant q^{\mathrm{gor}(M)(n - \min(m,n))} = |y|^{-\mathrm{gor}(M)} \|\boldsymbol{x}, y\|^{\mathrm{gor}(M)}. \qquad \blacklozenge$$

The above inequality is sharp (cf. the comments after Proposition 3.3):

Lemma 5.2. *Let $\boldsymbol{x} \in V$ and $y \in \mathfrak{O} \setminus \{0\}$. Then $\mathrm{O}_{\mathrm{M}_{d\times e}(\mathfrak{O})}(\boldsymbol{x}, y) = |y|^{-e} \|\boldsymbol{x}, y\|^{e}$.*

Proof. Let $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_d) \subset \mathfrak{O}^d$ be the standard basis. We may assume that $\boldsymbol{x} \neq 0$. As $\boldsymbol{x}\, \mathrm{M}_{d\times e}(\mathfrak{O})$ is generated by $\{\gcd(x_1, \dots, x_d)\boldsymbol{e}_i : i = 1, \dots, e\}$, the claim follows from Lemma 4.6(iv). $\blacklozenge$

Lemma 5.3. *The following are equivalent:*

(i) $\mathrm{O}_M(\boldsymbol{x}, y) = |y|^{-\mathrm{gor}(M)} \|\boldsymbol{x}, y\|^{\mathrm{gor}(M)}$ *for all $(\boldsymbol{x}, y) \in V \times \mathfrak{O}$ outside a set of measure zero.*

(ii) $\mathrm{O}_M(\boldsymbol{x}, y) = |y|^{-\mathrm{gor}(M)} \|\boldsymbol{x}, y\|^{\mathrm{gor}(M)}$ *for all $\boldsymbol{x} \in V$ and all $y \in \mathfrak{O} \setminus \{0\}$.*

Proof. O_M is locally constant on $V \times (\mathfrak{O} \setminus \{0\})$ so (i) implies (ii); the converse is clear. $\blacklozenge$

Definition 5.4. We say that M is **O-maximal** if it satisfies one of the two equivalent conditions in the preceding lemma.

Proposition 1.5 serves as a blueprint for $\mathsf{Z}_M(T)$ whenever M is O-maximal:

Corollary 5.5. *M is O-maximal if and only if $\mathsf{Z}_M(T) = \mathsf{Z}_{\mathrm{M}_{d\times \mathrm{gor}(M)}(\mathfrak{O})}(T)$.*

Proof. The "only if" part follows by combining (4.6) and Lemma 5.2. Conversely, suppose that $\mathrm{O}_M(\boldsymbol{x}, y) < |y|^{-\mathrm{gor}(M)} \|\boldsymbol{x}, y\|^{\mathrm{gor}(M)}$ for some $\boldsymbol{x} \in V$ and $y \in \mathfrak{O} \setminus \{0\}$. Using the fact that both sides of this inequality are locally constant functions of $(\boldsymbol{x}, y)$, Lemmas 5.1–5.2, and Theorem 4.5, we conclude that for sufficiently large $s \in \mathbf{R}$, $\zeta_M(s) > \zeta_{\mathrm{M}_{d\times \mathrm{gor}(M)}(\mathfrak{O})}(s)$. In particular, $\mathsf{Z}_M(T) \neq \mathsf{Z}_{\mathrm{M}_{d\times \mathrm{gor}(M)}(\mathfrak{O})}(T)$. ♦

The following is a "projective" characterisation of O-maximality.

Lemma 5.6. *M is* O*-maximal if and only if $\mathrm{O}_M(\boldsymbol{x}, y) = |y|^{-\mathrm{gor}(M)}$ for all $(\boldsymbol{x}, y) \in (V \setminus \mathfrak{P}V) \times \mathfrak{P}$ outside a set of measure zero.*

Proof. Necessity of the given condition being clear, suppose that $\mathrm{O}_M(\boldsymbol{x}, y) = |y|^{-\mathrm{gor}(M)}$ for all $(\boldsymbol{x}, y) \in ((V \setminus \mathfrak{P}V) \times \mathfrak{P}) \setminus Z$, where $Z \subset V \times \mathfrak{O}$ has measure zero. Choose $\pi \in \mathfrak{P} \setminus \mathfrak{P}^2$. For each $n \geqslant 0$, we recursively define a set $Z^{(n)} \subset V \times \mathfrak{O}$ of measure zero such that $\mathrm{O}_M(\pi^n \boldsymbol{x}, y) = |y|^{-\mathrm{gor}(M)} \|\pi^n, y\|^{\mathrm{gor}(M)}$ for all $(\boldsymbol{x}, y) \in ((V \setminus \mathfrak{P}V) \times \mathfrak{O}) \setminus Z^{(n)}$. By assumption and since $\mathrm{O}_M(\boldsymbol{x}, y) = 1$ for all $\boldsymbol{x} \in V$ and $y \in \mathfrak{O}^\times$, we may take $Z^{(0)} := Z$. Suppose that $Z^{(n)}$ has been defined with the aforementioned properties and let $Z^{(n+1)} := \{(\boldsymbol{x}, y) \in V \times \mathfrak{O} : (\boldsymbol{x}, \pi^{-1} y) \in Z^{(n)}\}$; note that $Z^{(n+1)}$ has measure zero. Let $(\boldsymbol{x}, y) \in ((V \setminus \mathfrak{P}V) \times \mathfrak{O}) \setminus Z^{(n+1)}$. We may assume that $y \in \mathfrak{P}$, say $y = \pi z$. Then, since $(\boldsymbol{x}, z) \notin Z^{(n)}$, using Lemma 4.16(i), we obtain

$$\begin{aligned}
\mathrm{O}_M(\pi^{n+1} \boldsymbol{x}, y) &= \mathrm{O}_M(\pi^n \boldsymbol{x}, z) \\
&= |z|^{-\mathrm{gor}(M)} \cdot \|\pi^n, z\|^{\mathrm{gor}(M)} \\
&= |y|^{-\mathrm{gor}(M)} \cdot \|\pi^{n+1}, y\|^{\mathrm{gor}(M)}.
\end{aligned}$$

Since $\{(\pi^n \boldsymbol{x}, y) : n \geqslant 0, (\boldsymbol{x}, y) \in Z^{(n)}\}$ has measure zero, the claim follows. ♦

We will repeatedly use the following lemma to prove O-maximality.

Lemma 5.7. *Let $\boldsymbol{X} = (X_1, \dots, X_d)$. Let $C(\boldsymbol{X}) \in \mathrm{M}_{\ell \times e}(\mathfrak{O}[\boldsymbol{X}])$ with $\mathfrak{O}[\boldsymbol{X}]^d C(\boldsymbol{X}) = \boldsymbol{X} \cdot (M \otimes \mathfrak{O}[\boldsymbol{X}])$. Suppose that there exists an $N \geqslant 0$ such that for $i = 1, \dots, \mathrm{gor}(M)$, the ideal of $\mathfrak{O}[\boldsymbol{X}]$ generated by the $i \times i$ minors of $C(\boldsymbol{X})$ contains each of $X_1^N, \dots, X_d^N$. Then M is* O*-maximal.*

Proof. Let $Z = \{\boldsymbol{x} \in V : \dim_K(\boldsymbol{x}M \otimes K) < \mathrm{gor}(M)\}$ as in Corollary 4.14. Let $\boldsymbol{x} \in \mathfrak{O}^d \setminus (\mathfrak{P}^d \cup Z)$ and let $y \in \mathfrak{O} \setminus \{0\}$. As in §4.3.5, let $\boldsymbol{g}_i(\boldsymbol{X})$ be the set of non-zero $i \times i$ minors of $C(\boldsymbol{X})$. Then, by assumption and since $\|\boldsymbol{x}\| = 1$, we have $\|\boldsymbol{g}_i(\boldsymbol{x})\| = 1$ for $i = 0, \dots, \mathrm{gor}(M)$ whence $\mathrm{O}_M(\boldsymbol{x}, y) = |y|^{-\mathrm{gor}(M)}$ by Corollary 4.14. Thus, M is O-maximal by Lemma 5.6. ♦

For a geometric interpretation of Lemma 5.7 in a global setting, see Proposition 6.9.

5.2 Proof of Proposition 1.5

Our proof of Proposition 1.5 and other computations in §5.3 rely on the following.

Lemma 5.8. *Let $a_0, \dots, a_r \in \mathbf{C}$ and write $\sigma_j = a_0 + \dots + a_j$. Suppose that the integral*

$$F_r(a_0, \dots, a_r) := \int\limits_{\mathfrak{O}^r \times \mathfrak{O}} |y|^{a_0} \|x_1, y\|^{a_1} \cdots \|x_1, \dots, x_r, y\|^{a_r} \, \mathrm{d}\mu_{\mathfrak{O}^r \times \mathfrak{O}}(\boldsymbol{x}, y),$$

is absolutely convergent. Then

$$F_r(a_0, \dots, a_r) = \frac{1 - q^{-1}}{1 - q^{-\sigma_r - r - 1}} \cdot \prod_{j=0}^{r-1} \frac{1 - q^{-\sigma_j - j - 2}}{1 - q^{-\sigma_j - j - 1}}.$$

In particular, in the special case $a_1 = \dots = a_{r-1} = 0$, we obtain

$$F_r(a_0, 0, \dots, 0, a_r) = \frac{(1 - q^{-1})(1 - q^{-a_0 - r - 1})}{(1 - q^{-a_0 - a_r - r - 1})(1 - q^{-a_0 - 1})}.$$

Proof. Both claims follow by induction from the identities (a) $F_0(a_0) = \frac{1 - q^{-1}}{1 - q^{-1 - a_0}}$ and (b) $F_r(a_0, \dots, a_r) = F_{r-1}(a_0 + a_1 + 1, a_2, \dots, a_r) \cdot \frac{1 - q^{-a_0 - 2}}{1 - q^{-a_0 - 1}}$ for $r \geqslant 1$. The formula for $F_0(a_0)$ in (a) is well-known and easily proved. By performing a change of variables according to whether $|x_1| \leqslant |y|$ or $|x_1| > |y|$, we find that $F_r(a_0, \dots, a_r) = F_{r-1}(a_0 + a_1 + 1, a_2, \dots, a_r) \cdot (1 + \int_{\mathfrak{P}} |y|^{a_0} \, \mathrm{d}\mu_{\mathfrak{O}}(y))$ whence (b) follows readily. ♦

Proof of Proposition 1.5. Using (4.6) and Lemma 5.2, we obtain

$$\mathsf{Z}_{\mathrm{M}_{d \times e}(\mathfrak{O})}(q^{-s}) = (1 - q^{-1})^{-1} \int\limits_{\mathfrak{O}^d \times \mathfrak{O}} |y|^{s + e - d - 1} \|\boldsymbol{x}, y\|^{-e} \, \mathrm{d}\mu_{\mathfrak{O}^d \times \mathfrak{O}}(\boldsymbol{x}, y). \tag{5.1}$$

whence the claim follows from Lemma 5.8. ♦

Remark 5.9.

(i) Let $M \subset \mathrm{M}_{d \times e}(\mathfrak{O})$ be any submodule. By combining Lemma 5.1 and (5.1), we thus obtain another interpretation of the lower bound in Proposition 3.3 in the form $\alpha_M \geqslant \alpha_{\mathrm{M}_{d \times \mathrm{gor}(M)}(\mathfrak{O})} = \max(d - \mathrm{gor}(M), 0)$.

(ii) We note that (1.2) could also be derived in an elementary fashion (without using p-adic integration) using Lemma 2.1 and ad hoc computations with generating functions. Such an approach quickly becomes cumbersome for more complicated examples such as most of those in §9. The author is, moreover, unaware of elementary proofs of general results such as Theorems 1.4 and 4.18.

5.3 Classical Lie algebras and relatives

Reminder. Let R be a ring. Recall the definitions of the **special linear**, **orthogonal**, and **symplectic** Lie algebras

$$\begin{aligned}\mathfrak{sl}_d(R) &= \{a \in \mathfrak{gl}_d(R) : \mathrm{trace}(a) = 0\},\\ \mathfrak{so}_d(R) &= \{a \in \mathfrak{gl}_d(R) : a + a^\top = 0\} \quad \text{(assuming } \mathrm{char}(R) \neq 2\text{), and}\\ \mathfrak{sp}_{2d}(R) &= \left\{ \begin{bmatrix} a & b \\ c & -a^\top \end{bmatrix} : a, b, c \in \mathrm{M}_d(R), b = b^\top, c = c^\top \right\}.\end{aligned}$$

These are Lie subalgebras of $\mathfrak{gl}_d(R)$ and $\mathfrak{gl}_{2d}(R)$, respectively. Finally, we let $\mathfrak{tr}_d(R)$ and $\mathfrak{n}_d(R)$ denote the Lie subalgebras of $\mathfrak{gl}_d(R)$ consisting of upper triangular matrices and strictly upper triangular matrices, respectively.

We now determine $\mathsf{Z}_{\mathfrak{g}}(T)$, where $\mathfrak{g}$ is one of the Lie algebras from above. Of course, the case $\mathfrak{g} = \mathfrak{gl}_d(\mathfrak{O})$ is covered by Proposition 1.5. Next, clearly, $\mathsf{Z}_{\mathfrak{sl}_1(\mathfrak{O})}(T) = 1/(1 - qT)$. The general case of $\mathfrak{sl}_d(\mathfrak{O})$ offers nothing new.

Corollary 5.10. *Let $d > 1$. Then $\mathsf{Z}_{\mathfrak{sl}_d(\mathfrak{O})}(T) = \mathsf{Z}_{\mathfrak{gl}_d(\mathfrak{O})}(T) = \frac{1-q^{-d}T}{(1-T)^2}$.*

Proof. It suffices to show that $\boldsymbol{x} \cdot \mathfrak{sl}_d(\mathfrak{O}) \supset \boldsymbol{x} \cdot \mathfrak{gl}_d(\mathfrak{O})$ for all $\boldsymbol{x} \in \mathfrak{O}^d \setminus \{0\}$. Let x_ℓ have minimal valuation among the entries of $\boldsymbol{x}$. Let $\boldsymbol{e}_j \in \mathfrak{O}^d$ be the jth unit vector. By our proof of Lemma 5.2, $\boldsymbol{x} \cdot \mathfrak{gl}_d(\mathfrak{O})$ is generated by $\{x_\ell \boldsymbol{e}_j : 1 \leqslant j \leqslant d\}$. Note that $\boldsymbol{x} \cdot \mathfrak{sl}_d(\mathfrak{O})$ is spanned by all $x_i \boldsymbol{e}_j$ and $x_i \boldsymbol{e}_i - x_j \boldsymbol{e}_j$ for $1 \leqslant i, j \leqslant d$ with $i \neq j$. It thus only remains to show that $x_\ell \boldsymbol{e}_\ell \in \boldsymbol{x} \cdot \mathfrak{sl}_d(\mathfrak{O})$. Since $d > 1$, we may choose $j \neq \ell$. Since $|x_j| \leqslant |x_\ell|$, $x_\ell \boldsymbol{e}_\ell = (x_\ell \boldsymbol{e}_\ell - x_j \boldsymbol{e}_j) + \frac{x_j}{x_\ell} x_\ell \boldsymbol{e}_j \in \boldsymbol{x} \cdot \mathfrak{sl}_d(\mathfrak{O})$ whence the claim follows. ♦

Proposition 5.11. *Let $\mathrm{char}(K) \neq 2$. Then $\mathsf{Z}_{\mathfrak{so}_d(\mathfrak{O})}(T) = \mathsf{Z}_{\mathrm{M}_{d\times(d-1)}(\mathfrak{O})}(T) = \frac{1-q^{1-d}T}{(1-T)(1-qT)}$.*

Remark 5.12.

(i) It is instructive to first determine $\mathrm{ask}(\mathfrak{so}_d(\mathbf{F}_q))$ for odd q. If F is any field with $\mathrm{char}(F) \neq 2$, then it is easy to see that $\boldsymbol{x} \cdot \mathfrak{so}_d(F) = \boldsymbol{x}^\perp$ for all $\boldsymbol{x} \in F^d \setminus \{0\}$, where the orthogonal complement is taken with respect to the standard inner product. In particular, if $\boldsymbol{x} \neq 0$, then $\dim_F(\boldsymbol{x} \cdot \mathfrak{so}_d(F)) = d - 1$. Using Lemma 2.1, we conclude that $\mathrm{ask}(\mathfrak{so}_d(\mathbf{F}_q)) = 1 + \frac{q^d-1}{q^{d-1}} = 1 - q^{1-d} + q$ for odd q; this identity was first proved probabilistically by Fulman and Goldstein [36, Lem. 5.3]. We note that for $\mathrm{char}(K) \neq 2$ and $\boldsymbol{x} \in \mathfrak{O}^d$, while we still have an inclusion $\boldsymbol{x} \cdot \mathfrak{so}_d(\mathfrak{O}) \subset \boldsymbol{x}^\perp$, equality does not, in general, hold; indeed, $\boldsymbol{x}^\perp$ is always an isolated submodule of $\mathfrak{O}^d$.

(ii) While we assumed that $\mathrm{char}(K) \neq 2$ in Proposition 5.11, we do allow $\mathrm{char}(\mathfrak{O}/\mathfrak{P}) = 2$. Note, however, that in this case, $\mathfrak{so}_d(\mathfrak{O})_n (= \mathfrak{so}_d(\mathfrak{O}) \otimes \mathfrak{O}_n)$ is properly contained in the set of all skew-symmetric $d \times d$-matrices over $\mathfrak{O}_n$.

Proof of Proposition 5.11. Part (i) of the preceding remark implies that $\operatorname{gor}(\mathfrak{so}_d(\mathfrak{O})) = d-1$. Given elements $z_1, \dots, z_\ell$ of some ring, we recursively define an $\binom{\ell}{2} \times \ell$ matrix

$$m(z_1, \dots, z_\ell) := \left[\begin{array}{c|cccc} -z_2 & z_1 & & & \\ -z_3 & & z_1 & & \\ \vdots & & & \ddots & \\ -z_\ell & & & & z_1 \\ \hline 0 & & & & \\ \vdots & & m(z_2, \dots, z_\ell) & & \\ 0 & & & & \end{array}\right];$$

for instance, $m(z_1)$ is the 0×1-matrix and $m(z_1, z_2) = [-z_2\ z_1]$.

Let $e_{ij} \in \mathrm{M}_d(\mathfrak{O})$ be the elementary matrix with 1 in position (i, j) and zeros elsewhere. Then the $e_{ij} - e_{ji}$ for $1 \leqslant i < j \leqslant d$ generate $\mathfrak{so}_d(\mathfrak{O})$ as an $\mathfrak{O}$-module whence the rows of $m(X_1, \dots, X_d)$ span $\boldsymbol{X} \cdot \mathfrak{so}_d(\mathfrak{O}[\boldsymbol{X}])$. In other words, the matrix $m(X_1, \dots, X_d)$ plays the role of $C(\boldsymbol{X})$ in §4.3.5 for $M = \mathfrak{so}_d(\mathfrak{O})$.

By induction, we may assume that $\pm X_2^i, \dots, \pm X_d^i$ are $i \times i$ minors of $m(X_2, \dots, X_d)$ for all $1 \leqslant i \leqslant d-2$ so that $\pm X_1^j, \dots, \pm X_d^j$ are $j \times j$ minors of $m(X_1, \dots, X_d)$ for $1 \leqslant j \leqslant d-1$. Thus, $\mathfrak{so}_d(\mathfrak{O})$ is O-maximal by Lemma 5.7 and the claim follows from Corollary 5.5. ♦

For a ring R, let $\mathrm{Sym}_d(R) = \{a \in \mathrm{M}_d(R) : a^\top = a\}$.

Proposition 5.13. $\mathsf{Z}_{\mathrm{Sym}_d(\mathfrak{O})}(T) = \mathsf{Z}_{\mathfrak{gl}_d(\mathfrak{O})}(T) = \frac{1-q^{-d}T}{(1-T)^2}$.

Proof. This proof is similar to that of Proposition 5.11 and we use the same notation. By considering the images of the first unit vector in $\mathfrak{O}^d$ under the matrices e_{11} and $e_{1j} + e_{j1}$ $(2 \leqslant j \leqslant d)$, we find that $\operatorname{gor}(\mathrm{Sym}_d(\mathfrak{O})) = d$. Given $z_1, \dots, z_\ell$, recursively define an $\binom{\ell+1}{2} \times \ell$ matrix

$$m'(z_1, \dots, z_\ell) := \left[\begin{array}{c|cccc} z_1 & & & & \\ z_2 & z_1 & & & \\ \vdots & & \ddots & & \\ z_\ell & & & z_1 & \\ \hline 0 & & & & \\ \vdots & & m'(z_2, \dots, z_\ell) & & \\ 0 & & & & \end{array}\right].$$

An induction similar to the one in the proof of Proposition 5.11 shows that $X_1^i, \dots, X_d^i$ are $i \times i$ minors of $m'(X_1, \dots, X_d)$ for $1 \leqslant i \leqslant d$. As $\boldsymbol{X} \cdot \mathrm{Sym}_d(\mathfrak{O}[\boldsymbol{X}])$ is the row span of $m'(X_1, \dots, X_d)$ over $\mathfrak{O}[\boldsymbol{X}]$, the claim follows from Lemma 5.7 and Corollary 5.5. ♦

Proposition 5.14. $\mathsf{Z}_{\mathfrak{sp}_{2d}(\mathfrak{O})}(T) = \mathsf{Z}_{\mathfrak{gl}_{2d}(\mathfrak{O})}(T) = \frac{1-q^{-2d}T}{(1-T)^2}$.

Proof. We proceed along the same lines as the preceding two proofs. Let e_{ij} denote the usual elementary matrix, now of size $2d \times 2d$. Using these matrices, it is easy to see that $(1, 0, \dots, 0) \cdot \mathfrak{sp}_{2d}(\mathfrak{O}) = \mathfrak{O}^{2d}$ whence $\mathrm{gor}(\mathfrak{sp}_{2d}(\mathfrak{O})) = 2d$. As an $\mathfrak{O}$-module, $\mathfrak{sp}_{2d}(\mathfrak{O})$ is generated by the following matrices: (i) $e_{ij} - e_{d+j,d+i}$ $(1 \leqslant i, j \leqslant d)$, (ii) $e_{i,d+i}$, $e_{d+i,i}$ $(1 \leqslant i \leqslant d)$, and (iii) $e_{i,d+j} + e_{j,d+i}$, $e_{d+i,j} + e_{d+j,i}$ $(1 \leqslant i < j \leqslant d)$. Write $\boldsymbol{X} = (X_1, \dots, X_d)$ and $\boldsymbol{X}' = (X_1', \dots, X_d')$. Define $m'(z_1, \dots, z_\ell)$ as in the proof of Proposition 5.13. Then $(\boldsymbol{X}, \boldsymbol{X}') \cdot \mathfrak{sp}_{2d}(\mathfrak{O}[\boldsymbol{X}, \boldsymbol{X}'])$ is generated by the rows of

$$\tilde{m}(\boldsymbol{X}, \boldsymbol{X}') := \left[\begin{array}{ccc|ccc} X_1 & & & -X_1' & & \\ & \ddots & & \vdots & & \\ & & X_1 & -X_d' & & \\ \hline & \vdots & & & \ddots & \\ \hline X_d & & & & & -X_1' \\ & \ddots & & & & \vdots \\ & & X_d & & & -X_d' \\ \hline\hline & & & & m'(\boldsymbol{X}) & \\ \hline & m'(\boldsymbol{X}') & & & & \end{array}\right].$$

Using what we have shown about the minors of $m'(\boldsymbol{X})$ in the proof of Proposition 5.13, we conclude that X_i^j and $(X_i')^j$ $(i = 1, \dots, d; j = 1, \dots, 2d)$ arise as $j \times j$ minors of $\tilde{m}(\boldsymbol{X}, \boldsymbol{X}')$. Again, the claim thus follows from Lemma 5.7 and Corollary 5.5. ♦

In contrast to the above examples, neither $\mathfrak{n}_d(\mathfrak{O})$ nor $\mathfrak{tr}_d(\mathfrak{O})$ (for $d > 1$) is O-maximal.

Proposition 5.15.

(i) $\mathsf{Z}_{\mathfrak{n}_d(\mathfrak{O})}(T) = \frac{(1-T)^{d-1}}{(1-qT)^d}$.

(ii) $\mathsf{Z}_{\mathfrak{tr}_d(\mathfrak{O})}(T) = \mathsf{Z}_{\mathfrak{n}_{d+1}(\mathfrak{O})}(q^{-1}T) = \frac{(1-q^{-1}T)^d}{(1-T)^{d+1}}$.

Proof. Since $\mathfrak{n}_{d+1}(\mathfrak{O})$ is obtained from $\mathfrak{tr}_d(\mathfrak{O})$ by adding a zero row and a zero column, by Corollary 3.6, it suffices to prove (i). Let $\boldsymbol{x} \in \mathfrak{O}^d \setminus \{0\}$. Then $\boldsymbol{x}\,\mathfrak{n}_d(\mathfrak{O})$ is generated by

$$\Big\{(0, x_1, 0, \dots, 0),\ (0, 0, \gcd(x_1, x_2), 0, \dots, 0),\ \dots,\ (0, \dots, 0, \gcd(x_1, \dots, x_{d-1}))\Big\};$$

in particular, $\mathrm{gor}(\mathfrak{n}_d(\mathfrak{O})) = d - 1$. Moreover, by (4.6) and Lemma 5.2,

$$\mathrm{O}_{\mathfrak{n}_d(\mathfrak{O})}(\boldsymbol{x}, y) = |y|^{1-d} \cdot \|x_1, y\| \cdot \|x_1, x_2, y\| \cdots \|x_1, \dots, x_{d-1}, y\|.$$

Hence, using (4.6) and Lemma 5.8, $(1 - q^{-1})\mathsf{Z}_{\mathfrak{n}_d(\mathfrak{O})}(q^{-s}) = F_{d-1}(s - 2, -1, \dots, -1)$. ♦

5.4 Diagonal matrices

Let $\mathfrak{d}_d(\mathfrak{O}) \subset \mathfrak{gl}_d(\mathfrak{O})$ be the subalgebra of diagonal matrices. Clearly $\mathfrak{d}_1(\mathfrak{O}) = \mathfrak{gl}_1(\mathfrak{O})$ so that $\mathsf{Z}_{\mathfrak{d}_1(\mathfrak{O})}(T) = \frac{1-q^{-1}T}{(1-T)^2}$. It turns out that the functions $\mathsf{Z}_{\mathfrak{d}_d(\mathfrak{O})}(T)$ have essentially been computed by Brenti [11, Thm 3.4] in a different context; the author is grateful to Angela Carnevale for pointing this out. First recall the definitions of B_n, $\mathrm{Des}(\sigma)$, and $\mathrm{N}(\sigma)$ from §2.3. For $\sigma \in \mathrm{B}_n$, let $\mathrm{d_B}(\sigma) := \#\,\mathrm{Des}(\sigma)$; the function $\mathrm{d_B}$ is known as the "descent statistic". Define a polynomial $h_n(X,Y) := \sum\limits_{\sigma\in\mathrm{B}_n} X^{\mathrm{N}(\sigma)} Y^{\mathrm{d_B}(\sigma)}$.

Theorem 5.16 ([11, Thm 3.4(ii)])**.** $\sum\limits_{i=0}^{\infty}(i(X+1)+1)^n Y^i = \frac{h_n(X,Y)}{(1-Y)^{n+1}}$ *for* $n \geqslant 1$.

The following marks a departure from the simplicity of previous examples of $\mathsf{Z}_M(T)$.

Corollary 5.17. $\mathsf{Z}_{\mathfrak{d}_d(\mathfrak{O})}(T) = \frac{h_d(-q^{-1},T)}{(1-T)^{d+1}}$.

Proof. By Corollary 3.6, $\mathsf{Z}_{\mathfrak{d}_d(\mathfrak{O})}(T)$ is the dth Hadamard power of $\mathsf{Z}_{\mathfrak{d}_1(\mathfrak{O})}(T)$. Since $\mathsf{Z}_{\mathfrak{d}_1(\mathfrak{O})}(T) = \frac{1-q^{-1}T}{(1-T)^2} = \sum_{i=0}^{\infty}(1+i-iq^{-1})T^i$, the claim follows from Theorem 5.16. ◆

Example 5.18.

(i) $\mathsf{Z}_{\mathfrak{d}_2(\mathfrak{O})}(T) = \dfrac{1+T-4q^{-1}T+q^{-2}T+q^{-2}T^2}{(1-T)^3}$.

(ii) $\mathsf{Z}_{\mathfrak{d}_3(\mathfrak{O})}(T) = \dfrac{1+6q^{-2}T-12q^{-1}T+4T+T^2-q^{-3}T-4q^{-3}T^2+12q^{-2}T^2-6q^{-1}T^2-q^{-3}T^3}{(1-T)^4}$.

We note that permutation statistics have previously featured in explicit formulae for representation zeta functions [12,83]; see also [13,14].

6 Constant rank spaces

By a **constant rank space** over a field F, we mean a subspace $M \subset \mathrm{M}_{d\times e}(F)$ such that all non-zero elements of M have the same rank, say r; we then say that M has constant rank r. Such spaces have been studied extensively in the literature (see e.g. [4,9,53,86,96]), often in the context of vector bundles on projective space. A problem of particular interest is to find, for given d and r, the largest possible dimension of a subspace of $\mathrm{M}_d(\mathbf{C})$ of constant rank r. Apart from trivial examples such as band matrices (see Example 6.6 below), the construction of constant rank spaces (in particular those of large dimension) seems to be challenging. Note that if $M \subset \mathrm{M}_{d\times e}(\mathbf{F}_q)$ has constant rank r and dimension ℓ, then

$$\mathrm{ask}(M) = q^{-\ell}(q^d + (q^\ell - 1)q^{d-r}) = q^{d-\ell} + q^{d-r} - q^{d-\ell-r}. \tag{6.1}$$

In §6.1, we consider a natural analogue, K-minimality, of the concept of O-maximality studied in §5.1. We then derive interpretations of these notions in a global setting in §6.2—in particular, we will see that K-minimality is related to constant rank spaces.

6.1 K-minimality

Let $\mathfrak{O}$ be the valuation ring of a non-Archimedean local field K of arbitrary characteristic. Let $M \subset \mathrm{M}_{d\times e}(\mathfrak{O})$ be a submodule. Recall the definition of K_M from Definition 4.4.

Lemma 6.1. *Let $F\colon \mathfrak{O}^\ell \to M$ be an $\mathfrak{O}$-module isomorphism, $\boldsymbol{w} \in \mathfrak{O}^\ell$, and $y \in \mathfrak{O} \setminus \{0\}$. Then $\mathrm{K}_M(F(\boldsymbol{w}), y) \geqslant |y|^{\mathrm{grk}(M)-d}\|\boldsymbol{w}, y\|^{-\mathrm{grk}(M)}$.*

Proof. We may assume that $\boldsymbol{w} \neq 0$. Let $a := F(\boldsymbol{w})$ have equivalence type $(\lambda_1, \dots, \lambda_r)$ (see §4.3.1); of course, $r \leqslant \mathrm{grk}(M)$. Let $m := \min(\nu(w_1), \dots, \nu(w_\ell))$, $n = \nu(y)$, and $a := F(\boldsymbol{w})$. Then $m \leqslant \min(\nu(a_{ij}) : 1 \leqslant i, j \leqslant d) = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_r$ and Lemma 4.6(iii) shows that

$$\begin{aligned}\mathrm{K}_M(F(\boldsymbol{w}), y) &\geqslant q^{r\min(m,n)+(d-r)n} \geqslant q^{\mathrm{grk}(M)\min(m,n)+(d-\mathrm{grk}(M))n} \\ &= |y|^{\mathrm{grk}(M)-d}\|\boldsymbol{w}, y\|^{-\mathrm{grk}(M)}.\end{aligned}$$ ♦

The following is analogous to Lemma 5.3.

Lemma 6.2. *For an $\mathfrak{O}$-module isomorphism $F\colon \mathfrak{O}^\ell \to M$, the following are equivalent:*

(i) $\mathrm{K}_M(F(\boldsymbol{w}), y) = |y|^{\mathrm{grk}(M)-d}\|\boldsymbol{w}, y\|^{-\mathrm{grk}(M)}$ *for all $(\boldsymbol{w}, y) \in \mathfrak{O}^\ell \times \mathfrak{O}$ outside a set of measure zero.*

(ii) $\mathrm{K}_M(F(\boldsymbol{w}), y) = |y|^{\mathrm{grk}(M)-d}\|\boldsymbol{w}, y\|^{-\mathrm{grk}(M)}$ *for all $\boldsymbol{w} \in \mathfrak{O}^\ell$ and $y \in \mathfrak{O} \setminus \{0\}$.* ♦

Definition 6.3. We say that M is **K-minimal** if there exists an $\mathfrak{O}$-module isomorphism $F\colon \mathfrak{O}^\ell \to M$ which satisfies one of the equivalent conditions from the preceding lemma.

Clearly, if $\boldsymbol{x} \in \mathfrak{O}^n$ and $a \in \mathrm{GL}_n(\mathfrak{O})$, then $\|\boldsymbol{x}a\| = \|x\|$. We conclude that if the condition in the preceding definition is satisfied for some isomorphism $F\colon \mathfrak{O}^\ell \to M$, then it holds for all of them. Lemma 5.8 and arguments as in the proof of Corollary 5.5 now imply the following.

Proposition 6.4. *Let $r = \mathrm{grk}(M)$ and $\ell = \dim_K(M \otimes K)$. Then M is K-minimal if and only if*

$$\mathsf{Z}_M(T) = \frac{1 - q^{d-\ell-r}T}{(1 - q^{d-\ell}T)(1 - q^{d-r}T)}.$$ ♦

The following sufficient condition for K-minimality is proved similarly to Lemma 5.7.

Lemma 6.5. *Let $(a_1, \dots, a_\ell)$ be an $\mathfrak{O}$-basis of M. Suppose that there exists $N \geqslant 0$ such that for $1 \leqslant i \leqslant \mathrm{grk}(M)$, the ideal generated by the $i \times i$ minors of $X_1a_1 + \cdots + X_\ell a_\ell \in \mathrm{M}_{d\times e}(\mathfrak{O}[X_1, \dots, X_\ell])$ contains $X_1^N, \dots, X_\ell^N$. Then M is K-minimal.* ♦

Example 6.6 (Band matrices)**.** Let $r \geqslant 1$ and define

$$B_r = \left\{ \begin{bmatrix} x_1 & & \\ x_2 & \ddots & \\ \vdots & \ddots & x_1 \\ x_r & \vdots & x_2 \\ & \ddots & \vdots \\ & & x_r \end{bmatrix} : x_1, \dots, x_r \in \mathfrak{O} \right\} \subset \mathrm{M}_{(2r-1)\times r}(\mathfrak{O}).$$

By Lemma 6.5 and Proposition 6.4 (with $d = 2r-1$ and $\ell = r$), $\mathsf{Z}_{B_r}(T) = \frac{1-q^{-1}T}{(1-q^{r-1}T)^2}$.

6.2 A global interpretation

Henceforth, let k be a number field with ring of integers $\mathfrak{o}$; recall the notation from §3.3. Let $\bar{k}$ be an algebraic closure of k. Let $M \subset \mathrm{M}_{d\times e}(\mathfrak{o})$ be a submodule. The following can be proved similarly to Proposition 4.8 (and Proposition 4.13).

Lemma 6.7.

(i) $\max\limits_{a\in M} \mathrm{rk}_k(a) = \max\limits_{\bar{a}\in M\otimes_{\mathfrak{o}}\bar{k}} \mathrm{rk}_{\bar{k}}(\bar{a}) = \mathrm{grk}(M_v)$ *for all* $v \in \mathcal{V}_k$*.*

(ii) $\max\limits_{\boldsymbol{x}\in\mathfrak{o}^d} \dim_k(\boldsymbol{x}\cdot(M\otimes_{\mathfrak{o}} k)) = \max\limits_{\bar{\boldsymbol{x}}\in\bar{k}^d} \dim_{\bar{k}}(\bar{\boldsymbol{x}}\cdot(M\otimes_{\mathfrak{o}}\bar{k})) = \mathrm{gor}(M_v)$ *for all* $v \in \mathcal{V}_k$*.* ♦

Extending Definitions 3.2 and 4.7, we let $\mathrm{grk}(M)$ and $\mathrm{gor}(M)$ be the common number in (i) and (ii), respectively. We will now see that K-minimality is closely related to constant rank spaces.

Proposition 6.8.

(i) Let $(a_1, \dots, a_\ell) \subset M$ *be a* k*-basis of* $M \otimes_{\mathfrak{o}} k$*. Let* I_i *be the ideal of* $k[X_1, \dots, X_\ell]$ *generated by the* $i \times i$ *minors of* $X_1a_1 + \dots + X_\ell a_\ell$*. Then* $M \otimes_{\mathfrak{o}} \bar{k}$ *is a constant rank space if and only if there exists* $N \geqslant 0$ *such that* $X_1^N, \dots, X_\ell^N \in I_i$ *for* $i = 1, \dots, \mathrm{grk}(M)$*.*

(ii) If $M \otimes_{\mathfrak{o}} \bar{k}$ *is a constant rank space, then* M_v *is* K*-minimal for almost all* $v \in \mathcal{V}_k$*.*

Proof.

(i) Let $\mathbf{V}(I) \subset \bar{k}^\ell$ be the algebraic set corresponding to $I \lhd k[\boldsymbol{X}] := k[X_1, \dots, X_\ell]$ and let $r = \mathrm{grk}(M)$. Then $M \otimes_{\mathfrak{o}} \bar{k}$ is a constant rank space if and only if $\mathbf{V}(I_i) \subset \{0\}$ for $i = 1, \dots, r$ which, by Hilbert's Nullstellensatz, is equivalent to $\sqrt{I_i} \supset \langle X_1, \dots, X_r\rangle$ for $i = 1, \dots, r$.

(ii) This follows from (i) and Lemma 6.5; note that if $f \in \mathfrak{o}[\boldsymbol{X}]$ belongs to the $k[\boldsymbol{X}]$-ideal generated by $g_1, \dots, g_r \in \mathfrak{o}[\boldsymbol{X}]$, then f also belongs to the $\mathfrak{o}_v[\boldsymbol{X}]$-ideal generated by the g_i, at least for almost all $v \in \mathcal{V}_k$. ♦

In view of Lemma 2.1, we say that a subspace $M' \subset \mathrm{M}_{d\times e}(F)$ (where F is a field) has **constant orbit dimension** if all F-spaces $\boldsymbol{x}M'$ for $\boldsymbol{x} \in F^d \setminus \{0\}$ have the same dimension. The following counterpart of Proposition 6.8 is then proved in the same way.

Proposition 6.9.

(i) *Let $(a_1, \dots, a_\ell) \subset M$ be a k-basis of $M \otimes_{\mathfrak{o}} k$. Let J_i be the ideal of $k[\boldsymbol{X}] := k[X_1, \dots, X_d]$ generated by the $i \times i$ minors of*

$$\begin{bmatrix} \boldsymbol{X}a_1 \\ \vdots \\ \boldsymbol{X}a_\ell \end{bmatrix}.$$

Then $M \otimes_{\mathfrak{o}} \bar{k}$ has constant orbit dimension. if and only if there exists $N \geqslant 0$ such that $X_1^N, \dots, X_d^N \in J_i$ for $i = 1, \dots, \mathrm{gor}(M)$.

(ii) *If $M \otimes_{\mathfrak{o}} \bar{k}$ has constant orbit dimension, then M_v is* O*-maximal for almost all $v \in \mathcal{V}_k$.* ♦

7 Smooth determinantal hypersurfaces

In a series of papers, Voll [89–91] developed geometric techniques for studying the normal subgroup growth of finitely generated torsion-free nilpotent groups of class 2. Under suitable genericity assumptions on the Pfaffian hypersurface attached to such a group, he produced an explicit formula [89, Thm 3] for almost all of its local normal subgroup zeta functions in terms of numbers of rational points of the aforementioned hypersurface.

The following is an analogue of Voll's result for ask zeta functions. Here, the role of the Pfaffian hypersurface of a group is played by the determinantal hypersurface associated with a matrix of linear forms, a classical topic in algebraic geometry (see Remark 7.9). Throughout this section, K is a non-Archimedean local field of arbitrary characteristic with valuation ring $\mathfrak{O}$. Recall that $\mathfrak{K} = \mathfrak{O}/\mathfrak{P}$ denotes the residue field of K.

Theorem 7.1. *Let $a_1, \dots, a_\ell \in \mathrm{M}_d(\mathfrak{O})$, where $\ell \geqslant 1$. Let $\boldsymbol{X} = (X_1, \dots, X_\ell)$ consist of algebraically independent variables over $\mathfrak{O}$. Write $a(\boldsymbol{X}) = X_1a_1 + \dots + X_\ell a_\ell \in \mathrm{M}_d(\mathfrak{O}[\boldsymbol{X}])$ and let $M = \{a(\boldsymbol{x}) : \boldsymbol{x} \in \mathfrak{O}^\ell\} \subset \mathrm{M}_d(\mathfrak{O})$. Let $F(\boldsymbol{X}) := \det(a(\boldsymbol{X}))$. Suppose that the following smoothness condition is satisfied:*

$$\textit{For all } \bar{\boldsymbol{x}} \in \mathfrak{K}^\ell, \textit{ if } F(\bar{\boldsymbol{x}}) = \frac{\partial F(\bar{\boldsymbol{x}})}{\partial X_1} = \dots = \frac{\partial F(\bar{\boldsymbol{x}})}{\partial X_\ell} = 0, \textit{ then } \bar{\boldsymbol{x}} = 0. \qquad \text{(SM)}$$

Let $H := \mathrm{Proj}(\mathfrak{O}[\boldsymbol{X}]/(F(\boldsymbol{X}))) \subset \mathbf{P}^{\ell-1}_{\mathfrak{O}}$. Then:

$$\mathsf{Z}_M(T) = \frac{1 - q^{-\ell}T}{(1-T)(1-q^{d-\ell}T)} + \#H(\mathfrak{K}) \cdot (q-1)^2 \cdot \frac{q^{-\ell}T}{(1-T)^2(1-q^{d-\ell}T)}.$$

A proof of Theorem 7.1 will be given below. We henceforth use the notation of Theorem 7.1 and assume that condition (SM) is satisfied. Note that the latter assumption is certainly satisfied if H is smooth as a scheme over $\mathfrak{O}$.

Let $I_i(\boldsymbol{X})$ be the ideal of $\mathfrak{O}[\boldsymbol{X}]$ generated by the $i \times i$ minors of $a(\boldsymbol{X})$. For an ideal $J(\boldsymbol{X}) \triangleleft \mathfrak{O}[\boldsymbol{X}]$ and an element b of an (associative, commutative, unital) $\mathfrak{O}$-algebra B, we write $J(b) = \{f(b) : f \in J(\boldsymbol{X})\} \triangleleft B$.

Lemma 7.2 (Cf. [17, Pf of Thm 2.2])**.**

(i) $\dfrac{\partial F(\boldsymbol{X})}{\partial X_i} \in I_{d-1}(\boldsymbol{X})$ *for* $i = 1, \dots, \ell$.

(ii) Let $\bar{\boldsymbol{x}} \in \mathfrak{K}^\ell \setminus \{0\}$. *Then* $\operatorname{rk}_{\mathfrak{K}}(a(\bar{\boldsymbol{x}})) \in \{d-1, d\}$.

Proof. See [17, p. 426] for (i). For (ii), if $\operatorname{rk}_{\mathfrak{K}}(a(\bar{\boldsymbol{x}})) < d-1$ for $\bar{\boldsymbol{x}} \in \mathfrak{K}^\ell$, then $F(\bar{\boldsymbol{x}}) = 0$ and $I_{d-1}(\bar{\boldsymbol{x}}) = \{0\}$. Part (i) and (SM) then imply that $\bar{\boldsymbol{x}} = 0$. ♦

Corollary 7.3. *If* $d \geqslant 2$, *then* $\mathfrak{O}^\ell \to M$, $\boldsymbol{x} \mapsto a(\boldsymbol{x})$ *is an isomorphism of* $\mathfrak{O}$*-modules.*

Proof. Let $\boldsymbol{x} \in \mathfrak{O}^\ell$ with $a(\boldsymbol{x}) = 0$ and suppose that $\boldsymbol{x} \neq 0$. Choose $\pi \in \mathfrak{P} \setminus \mathfrak{P}^2$. Then $\boldsymbol{x} = \pi^m \boldsymbol{y}$ for some $m \geqslant 0$ and an element $\boldsymbol{y} \in \mathfrak{O}^\ell$ whose image in $\mathfrak{K}^\ell$ is non-zero. Then $a(\boldsymbol{y}) = 0$ but also $\operatorname{rk}_{\mathfrak{K}}(a(\boldsymbol{y}) \otimes \mathfrak{K}) \geqslant d-1 > 0$ by Lemma 7.2(ii), a contradiction. ♦

Proof of Theorem 7.1. First suppose that $d = 1$. Then $a(\boldsymbol{X}) = F(\boldsymbol{X})$ is linear. Moreover, condition (SM) implies that $M = \mathrm{M}_1(\mathfrak{O})$ and $\#H(\mathfrak{K}) = (q^{\ell-1}-1)/(q-1)$. The claim is now easily verified using a direct computation and Proposition 1.5.

Let $d \geqslant 2$. Write $U = \mathfrak{O}^\ell$ and fix $\pi \in \mathfrak{P} \setminus \mathfrak{P}^2$. Let $\boldsymbol{x} \in U \setminus \mathfrak{P}U$ and $y \in \mathfrak{P} \setminus \{0\}$. By Lemma 7.2(ii), for each $i = 1, \dots, d-1$, some $i \times i$ minor of $a(\boldsymbol{x})$ belongs to $\mathfrak{O}^\times$. Hence, $I_0(\boldsymbol{x}) = \ldots = I_{d-1}(\boldsymbol{x}) = \mathfrak{O}$. As $F(\boldsymbol{X}) \neq 0$, $\operatorname{grk}(M) = d$. Using Corollaries 4.9 and 7.3,

$$\mathrm{K}_M(a(\boldsymbol{x}), y) = \prod_{i=1}^{d} \frac{\|I_{i-1}(\boldsymbol{x})\|}{\|I_i(\boldsymbol{x}) \cup y I_{i-1}(\boldsymbol{x})\|} = \|F(\boldsymbol{x}), y\|^{-1}.$$

Write $t = q^{-s}$. By Proposition 4.17(ii),

$$(1 - q^{d-\ell} t) \cdot \mathsf{Z}_M(t) = 1 + (1-q^{-1})^{-1} \int\limits_{(U \setminus \mathfrak{P}U) \times \mathfrak{P}} \omega, \tag{7.1}$$

where we wrote ω as a shorthand for $|y|^{s-1} \|F(\boldsymbol{x}), y\|^{-1} \mathrm{d}\mu_{U \times \mathfrak{O}}(\boldsymbol{x}, y)$. In order to evaluate the integral in (7.1), we decompose the domain of integration into sets of the form $(\boldsymbol{x}_0 + \mathfrak{P}U) \times \mathfrak{P}$ for $\boldsymbol{x}_0 \in U \setminus \mathfrak{P}U$. If $F(\boldsymbol{x}_0) \not\equiv 0 \pmod{\mathfrak{P}}$, then clearly

$$\int\limits_{(\boldsymbol{x}_0 + \mathfrak{P}U) \times \mathfrak{P}} \omega = \int\limits_{(\boldsymbol{x}_0 + \mathfrak{P}U) \times \mathfrak{P}} |y|^{s-1} \, \mathrm{d}\mu_{U \times \mathfrak{O}}(\boldsymbol{x}, y) = (1-q^{-1}) A(q,t),$$

where $A(q,t) := q^{-\ell} t/(1-t)$. Suppose that $F(\boldsymbol{x}_0) \equiv 0 \pmod{\mathfrak{P}}$. Using (SM) and Hensel's lemma [10, Ch. III, §4.5, Cor. 2], a measure-preserving change of coordinates transforms the map induced by $F(\boldsymbol{X})$ on $\boldsymbol{x}_0 + \mathfrak{P}U$ into the map induced by X_1, say, on $\mathfrak{P}U$. Thus,

$$\int\limits_{(\boldsymbol{x}_0+\mathfrak{P}U)\times\mathfrak{P}} \omega = \int\limits_{\mathfrak{P}U\times\mathfrak{P}} \frac{|y|^{s-1}}{\|x_1, y\|}\,\mathrm{d}\mu_{U\times\mathfrak{O}}(\boldsymbol{x}, y) = q^{1-\ell}\cdot \int\limits_{\mathfrak{P}\times\mathfrak{P}} \frac{|y|^{s-1}}{\|x, y\|}\,\mathrm{d}\mu_{\mathfrak{O}\times\mathfrak{O}}(x, y)$$

$$= q^{1-\ell}\cdot \sum_{m,n=1}^{\infty} \mu_{\mathfrak{O}\times\mathfrak{O}}(\pi^m\mathfrak{O}^\times \times \pi^n\mathfrak{O}^\times)\cdot q^{n+\min(m,n)}t^n$$

$$= (1-q^{-1})^2\cdot q^{1-\ell} \sum_{m,n=1}^{\infty} q^{\min(m,n)-m}\cdot t^n.$$

By an elementary calculation,

$$\sum_{m,n=1}^{\infty} q^{\min(m,n)-m}\cdot t^n = \frac{t(1-q^{-1}t)}{(1-q^{-1})(1-t)^2}$$

and thus $\int\limits_{(\boldsymbol{x}_0+\mathfrak{P}U)\times\mathfrak{P}} \omega = (1-q^{-1})B(q,t)$, where $B(q,t) = q^{1-\ell}t\cdot(1-q^{-1}t)/(1-t)^2$.

The condition $F(\boldsymbol{x}_0) \equiv 0 \pmod{\mathfrak{P}}$ is equivalent to the image of $\boldsymbol{x}_0$ in $\mathbf{P}^{\ell-1}(\mathfrak{K})$ being contained in $H(\mathfrak{K})$. Therefore,

$$(1-q^{d-\ell}t)\mathsf{Z}_M(t) = 1 + (1-q^{-1})^{-1} \int\limits_{(U\setminus\mathfrak{P}U)\times\mathfrak{P}} \omega$$

$$= 1 + (q-1)\left(\#\mathbf{P}^{\ell-1}(\mathfrak{K}) - \#H(\mathfrak{K})\right)A(q,t) + (q-1)\#H(\mathfrak{K})B(q,t)$$

$$= 1 + (q-1)\#\mathbf{P}^{\ell-1}(\mathfrak{K})\cdot\frac{q^{-\ell}t}{1-t} + \#H(\mathfrak{K})(q-1)\big(B(q,t)-A(q,t)\big)$$

and the claim follows from $1+(1-q^{-\ell})\frac{t}{1-t} = \frac{1-q^{-\ell}t}{1-t}$ and $B(q,t)-A(q,t) = (q-1)\frac{q^{-\ell}t}{(1-t)^2}$. ♦

Remark 7.4. By the Chevalley-Warning Theorem, $\#H(\mathfrak{K}) = 0$ implies that $\ell \leqslant d$; see e.g. [81, Ch. 1, §2, Cor. 1]. Suppose that $\#H(\mathfrak{K}) = 0$. Then M is readily seen to be K-minimal (cf. the proof of Theorem 7.1) and $M \otimes K$ has constant rank d. The formula $\mathsf{Z}_M(T) = \frac{1-q^{-\ell}T}{(1-T)(1-q^{d-\ell}T)}$ given by Theorem 7.1 in this case agrees with Proposition 6.4.

Example 7.5 (Diagonal 2×2 matrices)**.** Let $a(\boldsymbol{X}) = \mathrm{diag}(X_1,\dots,X_\ell)$. Then condition ($\mathsf{SM}$) is satisfied if and only if $\ell \leqslant 2$. Using Theorem 7.1 with $\ell = 2$ and $\#H(\mathfrak{K}) = 2$, we recover the special case $d = 2$ of Corollary 5.17.

Example 7.6 (Univariate polynomials)**.**

(i) (Local case.) Let $d \geqslant 1$ and $f(X) = X^d + c_{d-1}X^{d-1} + \dots + c_1X + c_0 \in \mathfrak{O}[X]$. Let

$$b_f = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -c_0 & \dots & -c_{m-2} & -c_{d-1} \end{bmatrix} \in \mathrm{M}_d(\mathfrak{O})$$

be the companion matrix of $f(X)$. Let $a_f(X,Y) = X \cdot 1_d - Y \cdot b_f \in \mathrm{M}_d(\mathfrak{O}[X,Y])$, where 1_d denotes the $d \times d$ identity matrix. Then $\det(a_f(X,Y)) = Y^d \cdot f(Y^{-1}X)$ is the homogenisation of $f(X)$. Let $M_f(\mathfrak{O}) = \{a_f(x,y) : x, y \in \mathfrak{O}\} \subset \mathrm{M}_d(\mathfrak{O})$; note that $M_f(\mathfrak{O})$ has $\mathfrak{O}$-rank 2 and that $\mathrm{grk}(M_f(\mathfrak{O})) = d$.

We now assume that the image of $f(X)$ in $\mathfrak{K}[X]$ has no repeated roots in $\mathfrak{K}$. Then condition (SM) is satisfied for $a(X,Y) = a_f(X,Y)$ and $F(X,Y) = \det(a_f(X,Y))$. We may therefore apply Theorem 7.1 (with $\ell = 2$) to obtain

$$\mathsf{Z}_{M_f(\mathfrak{O})}(T) = \frac{(1-T)(1-q^{-2}T) + n_f(\mathfrak{K})(1-q^{-1})^2 T}{(1-T)^2(1-q^{d-2}T)}, \tag{7.2}$$

where $n_f(\mathfrak{K}) = \#\{x \in \mathfrak{K} : f(x) = 0\} \in \{0, 1, \ldots, d\}$.

If the image of $f(X)$ in $\mathfrak{K}[X]$ is irreducible, then we are in the special case $n_f(\mathfrak{K}) = \#H(\mathfrak{K}) = 0$ discussed in Remark 7.4. At the other extreme, if $f(X)$ splits into d linear factors over $\mathfrak{K}$ (which is possible if and only if $q \leqslant d$), then $n_f(\mathfrak{K}) = d$.

(ii) (Global case.) Let k be a number field. Let $f(X) = X^d + c_{d-1}X^{d-1} + \cdots + c_0 \in \mathbf{Z}[X]$ be the minimal polynomial of an integral primitive element of $k/\mathbf{Q}$. For a (rational) prime p, the reduction $f_p(X) \in \mathbf{F}_p[X]$ of $f(X)$ modulo p is separable if and only if p does not divide the discriminant $\mathrm{disc}(f(X))$ of $f(X)$. Let $M_f \subset \mathrm{M}_d(\mathbf{Z})$ be the module of matrices generated by the identity matrix 1_d and the companion matrix of $f(X)$. Then, using the notation from (i), we may identify $M_f \otimes_{\mathbf{Z}} \mathbf{Z}_p = M_f(\mathbf{Z}_p)$. In particular, if $p \nmid \mathrm{disc}(f(X))$, then we obtain a formula (7.2) for $\mathsf{Z}_{M_f \otimes_{\mathbf{Z}} \mathbf{Z}_p}(T)$ which depends on the number $n_f(\mathbf{F}_p)$ of roots of $f(X)$ in $\mathbf{F}_p$.

Remark 7.7. For sufficiently large primes p, Voll [90, Prop. 3] gave an explicit formula for almost all local normal subgroup zeta functions associated with an indecomposable $\mathfrak{D}^*$-group attached to a power of an irreducible polynomial $f(X)$ over $\mathbf{Q}$; for background on $\mathfrak{D}^*$-groups, see [47]. Voll's formula depends on the number of roots of $f(X)$ modulo p and has essentially the same shape as (7.2); we note that the matrix $\mathbf{B}$ in [90, Eqn (16)] (and in [47, §6]) plays the same role as $a_f(X,Y)$ above.

Example 7.8 ($Y^2 = X^3 - X$)**.** Let $E \subset \mathbf{P}^2_{\mathbf{Z}}$ be defined by the homogenisation of $Y^2 = X^3 - X$. Then $E \otimes_{\mathbf{Z}} F$ is an elliptic curve for every field F of characteristic distinct from 2. The curve $E \otimes_{\mathbf{Z}} \mathbf{Q}$ has been previously used to show that various group-theoretic counting problems exhibit arithmetically "wild" behaviour. In particular, using determinantal representation in the sense of the present section, du Sautoy [26] constructed a nilpotent group of Hirsch length 9 whose local normal subgroup zeta function at a prime p depends on the number $\#E(\mathbf{F}_p)$ of $\mathbf{F}_p$-rational points of E. The precise shapes of these local zeta functions were first determined by Voll [90]. Due to the "wild" behaviour of $\#E(\mathbf{F}_p)$ as a function of p (see [26, Pf of Thm 2.1]), du Sautoy's result disproved earlier predictions on the growth of (normal) subgroups of nilpotent groups. His construction has since been used to demonstrate that other group-theoretic counting problems can be "wild" (e.g. the enumeration of representations [93, Ex. 2.4] or of “descendants” [31]). We will now see that the present setting is no exception. Let

$$a(X,Y,Z) = \begin{bmatrix} Z & X & Y \\ X & Z & 0 \\ Y & 0 & X \end{bmatrix}$$

and $M = \{a(x,y,z) : x,y,z \in \mathbf{Z}\} \subset \mathrm{M}_3(\mathbf{Z})$. Then E is defined by $\det(a(X,Y,Z)) = 0$. Suppose that q is odd so that $F(X,Y,Z) := \det(a(X,Y,Z))$ satisfies condition (SM). By applying Theorem 7.1, we thus obtain

$$\mathsf{Z}_{M\otimes_{\mathbf{Z}}\mathfrak{O}}(T) = \frac{1-q^{-3}T}{(1-T)^2} + \#E(\mathfrak{K})\cdot(q-1)^2\frac{q^{-3}T}{(1-T)^3}.$$

In particular, Theorem 4.11 accurately reflects the general dependence of $\mathsf{Z}_{M_v}(T)$ on a place $v \in \mathcal{V}_k$ for a module of matrices M over the ring of integers $\mathfrak{o}$ of a number field k. However, just as in the study of zeta functions of groups, it is presently unclear if anything meaningful can be said about the varieties $V_i \otimes_{\mathfrak{o}} k$ “required” to produce formulae (4.4) as M varies over all modules of matrices over $\mathfrak{o}$.

Remark 7.9. Determinantal representations of projective hypersurfaces (i.e. representations of defining polynomials as determinants of matrices of linear forms) over the complex numbers (or over algebraically closed fields) have been studied extensively; see e.g. [25, §§4.1, 9.3], [5] and [59]. In particular, in the smooth case, only curves and cubic surfaces over $\mathbf{C}$ generically admit determinantal representations. Ishitsuka [56, Cor. 8.3] showed that over a local field (of arbitrary characteristic), every smooth plane cubic admits a determinantal representation. He further showed [56, Thm 1.1(i)] that the same is true of a positive proportion (measured by height) of smooth plane cubics over $\mathbf{Q}$.

Remark 7.10. Let k be a number field with ring of integers $\mathfrak{o}$; recall the notation from §3.3. Let $a(\boldsymbol{X}) = a_1X_1 + \dots + a_\ell X_\ell \in \mathrm{M}_d(\mathfrak{o}[\boldsymbol{X}])$ for $\ell \geqslant 1$ and $a_1,\dots,a_\ell \in \mathrm{M}_d(\mathfrak{o})$. Let $F(\boldsymbol{X}) := \det(a(\boldsymbol{X}))$ and $M := \{a(\boldsymbol{x}) : \boldsymbol{x} \in \mathfrak{o}^\ell\}$. Define $\mathsf{H} := \mathrm{Proj}(\mathfrak{o}[\boldsymbol{X}]/(F(\boldsymbol{X})))$ and suppose that $\mathsf{H}\otimes_{\mathfrak{o}} k$ is smooth over k. For almost all $v \in \mathcal{V}_k$, Theorem 7.1 then provides a formula for $\mathsf{Z}_{M_v}(T)$ in terms of $\#\mathsf{H}(\mathfrak{K}_v)$. In this special case, for almost all $v \in \mathcal{V}_k$, the functional equation in Theorem 4.18 follows immediately from the identity

$$\#\mathsf{H}(\mathfrak{K}_v)\Big|_{q_v\to q_v^{-1}} = q_v^{2-\ell}\cdot\#\mathsf{H}(\mathfrak{K}_v),$$

a consequence of the Weil conjectures applied to the smooth projective variety $\mathsf{H}\otimes_{\mathfrak{o}}\mathfrak{K}_v$; cf. [23, Pf of Thm 4].

8 Orbits and conjugacy classes of linear groups

In this section, we use p-adic Lie theory to relate ask, orbit-counting, and conjugacy class zeta functions. In §8.1, we recall properties of saturable pro-p groups and Lie algebras. In §8.2, we prove that orbit-counting zeta functions over $\mathbf{Z}_p$ are rational. In §8.3 we compare group stabilisers and Lie centralisers under suitable hypotheses and this allows us to deduce Theorem 1.6 in §8.4. Finally, in §8.5, we prove Theorem 1.7.

8.1 Reminder: saturable pro-p groups and Lie algebras

We briefly recall Lazard's [64] notion of (p-)saturability of groups and Lie algebras using González-Sánchez's [38] equivalent formulation.

Let $\mathfrak{g}$ be a Lie $\mathbf{Z}_p$-algebra whose underlying $\mathbf{Z}_p$-module is free of finite rank. A **potent filtration** of $\mathfrak{g}$ is a central series $\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots$ of ideals (i.e. $[\mathfrak{g}_i, \mathfrak{g}] \subset \mathfrak{g}_{i+1}$ for all i) with $\bigcap\limits_{i=1}^{\infty} \mathfrak{g}_i = 0$ and such that $[\mathfrak{g}_i,_{p-1} \mathfrak{g}] := [\mathfrak{g}_i, \underbrace{\mathfrak{g}, \dots, \mathfrak{g}}_{p-1}] := [\dots[[\mathfrak{g}_i, \mathfrak{g}], \mathfrak{g}], \dots, \mathfrak{g}] \subset p\mathfrak{g}_{i+1}$ for all $i \geqslant 1$. We say that $\mathfrak{g}$ is **saturable** if it admits a potent filtration.

Proposition 8.1. *(*[1, Prop. 2.3]*) Let $\mathfrak{O}$ be the valuation ring of a non-Archimedean local field $K \supset \mathbf{Q}_p$. Let $\mathfrak{e}(K/\mathbf{Q}_p)$ denote the ramification index of $K/\mathbf{Q}_p$. Let $\mathfrak{g}$ be a Lie $\mathfrak{O}$-algebra whose underlying $\mathfrak{O}$-module is free of finite rank. Let $m > \frac{\mathfrak{e}(K/\mathbf{Q}_p)}{p-1}$. Then $\mathfrak{g}^m$ ($= \mathfrak{P}^m \mathfrak{g}$) is saturable as a $\mathbf{Z}_p$-algebra.*

We note that, as before, subalgebras of an $\mathfrak{O}$-algebra are understood to be $\mathfrak{O}$-subalgebras; whenever we consider $\mathbf{Z}_p$-subalgebras, we will explicitly state as much.

Similarly to the case of Lie algebras, a torsion-free finitely generated pro-p group G which admits a central series $G = G_1 \geqslant G_2 \geqslant \cdots$ of closed subgroups with $\bigcap\limits_{i=1}^{\infty} G_i = 1$ and $[G_i,_{p-1} G] \leqslant G_{i+1}^p$ is **saturable**.

If $\mathfrak{g}$ is a saturable Lie $\mathbf{Z}_p$-algebra, then the underlying topological space of $\mathfrak{g}$ can be endowed with the structure of a saturable pro-p group using the Hausdorff series. Conversely, every saturable pro-p group gives rise to a saturable Lie $\mathbf{Z}_p$-algebra and these two functorial operations furnish mutually quasi-inverse equivalences between the categories of saturable Lie $\mathbf{Z}_p$-algebras and saturable pro-p groups (defined as full subcategories of all Lie $\mathbf{Z}_p$-algebras and pro-p groups, respectively); see [38, §4] for an overview and [64, Ch. 4] for details. While the general interplay between subalgebras and subgroups is subtle, we note the following fact.

Lemma 8.2. *Let $\mathfrak{g}$ be a saturable Lie $\mathbf{Z}_p$-algebra and let $\mathfrak{h}$ be a saturable subalgebra of finite additive index. Let G and H be the saturable pro-p groups associated with $\mathfrak{g}$ and $\mathfrak{h}$ via the Hausdorff series (so that $H \leqslant G$). Then $|G : H| = |\mathfrak{g} : \mathfrak{h}|$.*

Proof. This can be proved in the same way as [39, Lem. 3.2(4)]. ♦

8.2 Orbits of p-adic linear groups

Let $\mathfrak{O}$ be the valuation ring of a non-Archimedean local field K. Recall that $\mathfrak{P}$ denotes the maximal ideal of $\mathfrak{O}$ and that q and p denote the size and characteristic of the residue field of K, respectively. Further recall the definition of $\mathsf{Z}_G^{\mathsf{oc}}(T)$ (Definition 1.2(ii)). Although we will not need it in the sequel, since it might be of independent interest, we note the following rationality statement for $\mathsf{Z}_G^{\mathsf{oc}}(T)$.

Theorem 8.3. *Let $G \leqslant \mathrm{GL}_d(\mathbf{Z}_p)$. Then $\mathsf{Z}_G^{\mathsf{oc}}(T) \in \mathbf{Q}(T)$. More precisely, there are $a_1, \dots, a_r \in \mathbf{Z}$ and $b_1, \dots, b_r \in \mathbf{N}$ such that $\prod_{i=1}^{r}(1 - p^{a_i}T^{b_i})\mathsf{Z}_G^{\mathsf{oc}}(T) \in \mathbf{Q}[T]$.*

Remark 8.4.

(i) The author does not know if the conclusion of Theorem 8.3 remains valid if G is allowed to be a subgroup of $\mathrm{GL}_d(\mathbf{F}_q[\![z]\!])$. The proof of Theorem 8.3 below combines basic p-adic Lie theory and a powerful model-theoretic result due to Cluckers [51, App. A]. Both of these ingredients are only available in characteristic zero. In the latter case, this reflects the mysterious nature of the model theory of local fields of (small) positive characteristic.

(ii) Avni et al. [2, Thms E, A.5] gave formulae for the "similarity class zeta functions" associated with the groups $\mathrm{GL}_d(\mathfrak{O})$ and $\mathrm{GU}_d(\mathfrak{O})$ for $d = 2, 3$. In addition to being consistent with Theorem 8.3, their formulae are valid for all local fields K subject to the sole assumption that q be odd in case of $\mathrm{GU}_d(\mathfrak{O})$. As explained in (i), the techniques employed here seem incapable of establishing rationality results in such great generality.

Lemma 8.5. *Let $\bar{G}$ be the closure of $G \leqslant \mathrm{GL}_d(\mathfrak{O})$ in $\mathrm{GL}_d(\mathfrak{O})$. Then $\mathsf{Z}^{\mathsf{oc}}_{\bar{G}}(T) = \mathsf{Z}^{\mathsf{oc}}_{G}(T)$.*

Proof. The open subgroups $\Gamma_i := \{a \in \mathrm{GL}_d(\mathfrak{O}) : a \equiv 1 \pmod{\mathfrak{P}^i}\}$ form a fundamental system of neighbourhoods of the identity in $\mathrm{GL}_d(\mathfrak{O})$. The claim follows since $G\Gamma_i = \bar{G}\Gamma_i$ for all $i \geqslant 0$. ♦

Proof of Theorem 8.3. Define an equivalence relation $\sim_n$ on $\mathbf{Z}_p^d$ via

$$\boldsymbol{x} \sim_n \boldsymbol{y} \quad :\Leftrightarrow \quad \exists g \in G.\, \boldsymbol{x} \equiv \boldsymbol{y}g \pmod{p^n}.$$

Our theorem will follow immediately from [51, Thm A.2] once we have established that $\sim_n$ is definable (definably in n) in the subanalytic language used in [51, App. A]. By the preceding lemma, we may assume that $G = \bar{G}$. It then follows from the well-known structure theory of p-adic analytic groups (see [24,64]) that there exists an open saturable (or, more restrictively, uniform) subgroup $H \leqslant G$ of the form $H = \exp(p^2\mathfrak{h})$, where $\mathfrak{h} \subset \mathfrak{gl}_d(\mathbf{Z}_p)$ is a suitable saturable (or uniform) $\mathbf{Z}_p$-subalgebra. Let T be a transversal for the right cosets of H in G. Then, for $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{Z}_p^d$, $\boldsymbol{x} \sim_n \boldsymbol{y}$ if and only if

$$\bigvee_{t \in T} \exists a \in \mathfrak{h}.\, \boldsymbol{x}\exp(p^2 a) \equiv \boldsymbol{y}t^{-1} \pmod{p^n}.$$

The claim thus follows since the $d \times d$-matrix exponential $\exp(p^2 X)$ is given by d^2 power series in d^2 variables which all converge on $\mathfrak{gl}_d(\mathbf{Z}_p)$. ♦

Remark 8.6. Let $K/\mathbf{Q}_p$ be a finite extension. Then we may regard $G \leqslant \mathrm{GL}_d(\mathfrak{O})$ as a $\mathbf{Z}_p$-linear group of degree $d|K : \mathbf{Q}_p|$ via the regular representation of $\mathfrak{O}$ over $\mathbf{Z}_p$. In particular, Theorem 8.3 implies that $\mathsf{Z}^{\mathsf{oc}}_{G}(T)$ is rational provided that $K/\mathbf{Q}_p$ is unramified.

The following questions are inspired by Theorems 4.11–4.12 and [6, Thm C].

Question 8.7. Let k be a number field with ring of integers $\mathfrak{o}$. Let $\mathbf{G} \leqslant \mathrm{GL}_d \otimes k$ be an algebraic group over k. Let G denote the schematic closure of $\mathbf{G}$ in $\mathrm{GL}_d \otimes \mathfrak{o}$.

(i) Do there exist $V_1, \dots, V_r$ and $W_1, \dots, W_r \in \mathbf{Q}(X, T)$ as in Theorem 4.11 such that

$$\mathsf{Z}^{\mathsf{oc}}_{\mathsf{G}(\mathfrak{o}_v)}(T) = \sum_{i=1}^{r} \#V_i(\mathfrak{K}_v) \cdot W_i(q_v, T)$$

for almost all places $v \in \mathcal{V}_k$?

(ii) Do we have $\mathsf{Z}^{\mathsf{oc}}_{\mathsf{G}(\mathfrak{o}_v)}(T) = \mathsf{Z}^{\mathsf{oc}}_{\mathsf{G}(\mathfrak{K}_v[\![z]\!])}(T)$ for almost all $v \in \mathcal{V}_k$?

By combining Theorem 4.11 and Corollary 8.19 below, we see that Question 8.7(i) has a positive answer if $\mathbf{G}$ is unipotent. We conclude this subsection with some elementary examples of $\mathsf{Z}^{\mathsf{oc}}_G(T)$.

Example 8.8.

(i) It is easy to see that the rule $\boldsymbol{x} \mapsto \min(\nu(x_1), \dots, \nu(x_d), n)$ induces a bijection $\mathfrak{O}_n^d / \operatorname{GL}_d(\mathfrak{O}) \to \{0, \dots, n\}$ whence

$$\mathsf{Z}^{\mathsf{oc}}_{\operatorname{GL}_d(\mathfrak{O})}(T) = \sum_{n=0}^{\infty} (n+1) T^n = \frac{1}{(1-T)^2}.$$

(ii) Let $p \neq 2$. The number of orbits of $\langle -1 \rangle \leqslant \operatorname{GL}_1(\mathfrak{O})$ on $\mathfrak{O}_n$ is $1 + (q^n - 1)/2$ so that

$$\mathsf{Z}^{\mathsf{oc}}_{\langle -1 \rangle}(T) = \frac{1}{1-T} + \frac{1}{2}\Big(\frac{1}{1-qT} - \frac{1}{1-T}\Big) = \frac{2 - qT - T}{2(1-qT)(1-T)}.$$

(iii) Let $G = \langle [\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}] \rangle \leqslant \operatorname{GL}_2(\mathfrak{O})$. It is easy to see that

$$\mathsf{Z}^{\mathsf{oc}}_G(T) = \sum_{n=0}^{\infty} \frac{q^n(q^n+1)}{2} T^n = \frac{2 - q^2 T - qT}{2(1-q^2T)(1-qT)}.$$

We note that the fact that the preceding examples as well as the formula in [2, (1.12)] all satisfy functional equations under the operation "$(q, T) \to (q^{-1}, T^{-1})$" does not seem to be explained by any of the results in the present article (e.g. Theorem 4.18).

8.3 Lie centralisers and group stabilisers

Let $\mathfrak{O}$ be the valuation ring of a local field $K \supset \mathbf{Q}_p$. Let $\mathfrak{e}(K/\mathbf{Q}_p)$ denote the ramification index of $K/\mathbf{Q}_p$. As expected, for suitable matrix algebras and groups, the equivalence between saturable pro-p groups and Lie $\mathbf{Z}_p$-algebras recalled in §8.1 can be made explicit using exponentials and logarithms.

In line with our previous notation (see §3.5), we write $\mathfrak{gl}_d^m(\mathfrak{O}) := \{a \in \mathfrak{gl}_d(\mathfrak{O}) : a \equiv 0 \pmod{\mathfrak{P}^m}\}$. Moreover, we write $\operatorname{GL}_d^m(\mathfrak{O}) := \{a \in \operatorname{GL}_d(\mathfrak{O}) : a \equiv 1 \pmod{\mathfrak{P}^m}\}$.

Proposition 8.9 ([60, Lem. B.1])**.** *Let $m > \frac{\mathfrak{e}(K/\mathbf{Q}_p)}{p-1}$. The formal exponential and logarithm series converge on $\mathfrak{gl}_d^m(\mathfrak{O})$ and $\operatorname{GL}_d^m(\mathfrak{O})$, respectively, and define mutually inverse bijections* $\exp\colon \mathfrak{gl}_d^m(\mathfrak{O}) \to \operatorname{GL}_d^m(\mathfrak{O})$ *and* $\log\colon \operatorname{GL}_d^m(\mathfrak{O}) \to \mathfrak{gl}_d^m(\mathfrak{O})$.

Hence, if $m > \frac{\mathfrak{e}(K/\mathbf{Q}_p)}{p-1}$ and $\mathfrak{g} \subset \mathfrak{gl}_d^m(\mathfrak{O})$ is a saturable subalgebra, then we may identify the saturable pro-p group associated with $\mathfrak{g}$ in the sense of §8.1 with $\exp(\mathfrak{g}) \leqslant \mathrm{GL}_d^m(\mathfrak{O})$.

For $\boldsymbol{x} \in \mathfrak{O}^d$ and $n \geqslant 0$, we write $\boldsymbol{x} \bmod \mathfrak{P}^n$ for the image of $\boldsymbol{x}$ in $\mathfrak{O}_n^d$.

Lemma 8.10. *Let $\mathfrak{g} \subset \mathfrak{gl}_d^{\mathfrak{e}(K/\mathbf{Q}_p)}(\mathfrak{O})$ $(= p\,\mathfrak{gl}_d(\mathfrak{O}))$ be a saturable subalgebra. Then*

$$\mathrm{c}_{\mathfrak{g}}(\boldsymbol{x} \bmod \mathfrak{P}^n) = \{a \in \mathfrak{g} : \boldsymbol{x}a \equiv 0 \pmod{\mathfrak{P}^n}\}$$

is a saturable subalgebra of $\mathfrak{g}$ for all $\boldsymbol{x} \in \mathfrak{O}^d$.

Proof. Write $\mathfrak{e} = \mathfrak{e}(K/\mathbf{Q}_p)$ and $\mathfrak{c}^n := \mathrm{c}_{\mathfrak{g}}(\boldsymbol{x} \bmod \mathfrak{P}^n)$; obviously, $\mathfrak{c}^n$ is a subalgebra of $\mathfrak{g}$. Let $\mathfrak{g} = \gamma_1(\mathfrak{g}) \supset \gamma_2(\mathfrak{g}) \supset \cdots$ be the lower central series of $\mathfrak{g}$. Then $\gamma_p(\mathfrak{c}^n) \subset \mathfrak{c}^{n+\mathfrak{e}}$. Indeed, each element, a say, of $\gamma_p(\mathfrak{c}^n)$ is a sum of matrix products $c(ph)$ for suitable $c \in \mathfrak{c}^n$ and $h \in \mathfrak{gl}_d(\mathfrak{O})$ and clearly, $\boldsymbol{x}c(ph) \equiv 0 \pmod{\mathfrak{P}^{n+\mathfrak{e}}}$. Let $\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots$ be a potent filtration of $\mathfrak{g}$. Then $\mathfrak{c}^n = \mathfrak{c}^n \cap \mathfrak{g}_1 \supset \mathfrak{c}^n \cap \mathfrak{g}_2 \supset \cdots$ is a central series of $\mathfrak{c}^n$ with $\bigcap\limits_{i=1}^{\infty} (\mathfrak{c}^n \cap \mathfrak{g}_i) = 0$. It is in fact a potent filtration for $p\mathfrak{g} \cap \mathfrak{c}^{n+\mathfrak{e}} = p\mathfrak{c}^n$ whence $[(\mathfrak{c}^n \cap \mathfrak{g}_i),_{p-1} \mathfrak{c}^n] \subset p\mathfrak{g}_{i+1} \cap \gamma_p(\mathfrak{c}^n) \subset p\mathfrak{g} \cap \mathfrak{c}^{n+\mathfrak{e}} = p\mathfrak{c}^n$. ♦

Lemma 8.11. *Let $m > \frac{\mathfrak{e}(K/\mathbf{Q}_p)}{p-1}$ and $a \in \mathfrak{gl}_d^m(\mathfrak{O})$. Then there exists $u \in \mathrm{GL}_d^1(\mathfrak{O})$ with $\exp(a) = 1 + au$.*

Proof. Let $g(X) := \sum_{i=0}^{\infty} \frac{1}{(i+1)!} X^i$ so that $\exp(X) = 1 + Xg(x)$ in $\mathbf{Q}[\![X]\!]$. Let $a \in \mathfrak{gl}_d^m(\mathfrak{O})$ be non-zero. Let b be an entry of a of minimal valuation. Then $\nu_K(b^i/(i+1)!)$ is a lower bound for the valuation of each entry of $\frac{1}{(i+1)!}a^i$ for $i \geqslant 0$. It is well-known that $\nu_p((i+1)!) \leqslant i/(p-1)$ (see e.g. [16, Lem. 4.2.8(1)]) so that $\nu_K((i+1)!) \leqslant \mathfrak{e}(K/\mathbf{Q}_p)i/(p-1)$. Therefore, $\nu_K(b^i/(i+1)!) \geqslant i(\nu_K(b) - \mathfrak{e}(K/\mathbf{Q}_p)/(p-1)) =: f(i)$. Clearly, $f(i) \to \infty$ as $i \to \infty$ and $f(i) > 0$ for $i > 0$. Hence, the series $g(a)$ converges in $\mathrm{M}_d(\mathfrak{O})$ to an element of $\mathrm{GL}_d^1(\mathfrak{O})$. ♦

By combining the preceding two lemmas, we obtain the following.

Corollary 8.12. *Let $m > \max\left(\mathfrak{e}(K/\mathbf{Q}_p) - 1, \frac{\mathfrak{e}(K/\mathbf{Q}_p)}{p-1}\right)$, let $\mathfrak{g} \subset \mathfrak{gl}_d^m(\mathfrak{O})$ be a saturable subalgebra, and let $\boldsymbol{x} \in \mathfrak{O}^d$. Then $\exp(\mathrm{c}_{\mathfrak{g}}(\boldsymbol{x} \bmod \mathfrak{P}^n)) = \mathrm{St}_{\exp(\mathfrak{g})}(\boldsymbol{x} \bmod \mathfrak{P}^n)$.*

Proof. Let $a \in \mathfrak{g}$ and write $\exp(a) = 1 + au$ for $u \in \mathrm{GL}_d^1(\mathfrak{O})$. Then

$$\begin{aligned} a \in \mathrm{c}_{\mathfrak{g}}(\boldsymbol{x} \bmod \mathfrak{P}^n) &\Leftrightarrow \boldsymbol{x}a \equiv 0 \pmod{\mathfrak{P}^n} \\ &\Leftrightarrow \boldsymbol{x}au \equiv 0 \pmod{\mathfrak{P}^n} \\ &\Leftrightarrow \boldsymbol{x}\exp(a) \equiv \boldsymbol{x}(1+au) \equiv \boldsymbol{x} \pmod{\mathfrak{P}^n} \\ &\Leftrightarrow \exp(a) \in \mathrm{St}_{\exp(\mathfrak{g})}(\boldsymbol{x} \bmod \mathfrak{P}^n). \end{aligned}$$

♦

8.4 Proof of Theorem 1.6

Let $\mathfrak{O}$ be the valuation ring of a local field $K \supset \mathbf{Q}_p$. Recall that $\mathfrak{e}(K/\mathbf{Q}_p)$ denotes the ramification index of $K/\mathbf{Q}_p$.

Proposition 8.13. *Let $m > \max\left(\mathfrak{e}(K/\mathbf{Q}_p) - 1, \frac{\mathfrak{e}(K/\mathbf{Q}_p)}{p-1}\right)$, let $\mathfrak{g} \subset \mathfrak{gl}_d^m(\mathfrak{O})$ be a saturable subalgebra, and let $G = \exp(\mathfrak{g})$. Then $\mathsf{Z}_{\mathfrak{g}}(T) = \mathsf{Z}_G^{\mathsf{oc}}(T)$.*

Proof. Write $V = \mathfrak{O}^d$ and $V_n = V \otimes \mathfrak{O}_n$. Recall that $\mathfrak{g}_n$ denotes the image of $\mathfrak{g}$ under the natural map $\mathfrak{gl}_d(\mathfrak{O}) \to \mathfrak{gl}_d(\mathfrak{O}_n)$. For $n \geqslant 0$, by combining Lemma 2.1, Lemma 8.2, and Corollary 8.12, we obtain

$$\begin{aligned}
|V_n/G| &= \sum_{\boldsymbol{x} \in V_n} |\boldsymbol{x}G|^{-1} = \sum_{\boldsymbol{x} \in V_n} |G : \mathrm{St}_G(\boldsymbol{x})|^{-1} \\
&= \sum_{\boldsymbol{x} \in V_n} |\mathfrak{g} : \mathrm{c}_{\mathfrak{g}}(\boldsymbol{x})|^{-1} = \sum_{\boldsymbol{x} \in V_n} |\mathfrak{g}_n : \mathrm{c}_{\mathfrak{g}_n}(\boldsymbol{x})|^{-1} \\
&= \sum_{\boldsymbol{x} \in V_n} |\boldsymbol{x}\,\mathfrak{g}_n|^{-1} = \mathrm{ask}(\mathfrak{g}_n \mid V_n).
\end{aligned}$$

♦

Proof of Theorem 1.6. Let $m > \max\left(\mathfrak{e}(K/\mathbf{Q}_p) - 1, \frac{\mathfrak{e}(K/\mathbf{Q}_p)}{p-1}\right)$. Propositions 3.9 and 8.13 show that $q^{dm}\mathsf{Z}_{\mathfrak{g}}(T) = \mathsf{Z}^m_{\mathfrak{g}^m}(T) = \mathsf{Z}^{\mathsf{oc},m}_{\exp(\mathfrak{g}^m)}(T) \in \mathbf{Z}[\![T]\!]$. ♦

8.5 Orbits and conjugacy classes of unipotent groups

Let $\mathfrak{O}$ be the valuation ring of a local field $K \supset \mathbf{Q}_p$. Recall that $\mathfrak{n}_d(\mathfrak{O}) \subset \mathfrak{gl}_d(\mathfrak{O})$ is the Lie algebra of all strictly upper triangular matrices and that U_d denotes the group scheme of upper unitriangular $d \times d$ matrices. The following is well-known.

Proposition 8.14. *Let $p \geqslant d$.*

(i) All $\mathbf{Z}_p$-subalgebras of $\mathfrak{n}_d(\mathfrak{O})$ and closed subgroups of $\mathrm{U}_d(\mathfrak{O})$ are saturable.

(ii) exp *and* log *define polynomial bijections between $\mathfrak{n}_d(\mathfrak{O})$ and $\mathrm{U}_d(\mathfrak{O})$.*

Proof. Noting that subalgebras of $\mathfrak{n}_d(\mathfrak{O})$ and closed subgroups of $\mathrm{U}_d(\mathfrak{O})$ are nilpotent of class at most $d-1 < p$, their lower central series constitute potent filtrations. This proves (i). Part (ii) follows since $a^d = 0$ for $a \in \mathfrak{n}_d(\mathfrak{O})$. ♦

A simple variation of Proposition 8.14 yields the following.

Corollary 8.15. *Let $\mathfrak{g} \subset \mathfrak{gl}_d(\mathfrak{O})$ be a subalgebra. Suppose that $\mathfrak{g}$ is nilpotent of class at most c and that $a^{c+1} = 0$ (in $\mathrm{M}_d(\mathfrak{O})$) for all $a \in \mathfrak{g}$. Further suppose that $p > c$.*

(i) $\mathfrak{g}$ is saturable.

(ii) $G := \exp(\mathfrak{g})$ is a saturable subgroup of $\mathrm{GL}_d(\mathfrak{O})$.

(iii) exp *and* log *define mutually inverse polynomial bijections between $\mathfrak{g}$ and G.* ♦

By going through the proof of Theorem 1.6, we now easily obtain the following.

Corollary 8.16. *Let the notation be as in Corollary 8.15. Then* $\mathsf{Z}^{\mathsf{ask}}_{\mathfrak{g}}(T) = \mathsf{Z}^{\mathsf{oc}}_{\exp(\mathfrak{g})}(T)$ *and thus, in particular,* $\mathsf{Z}^{\mathsf{ask}}_{\mathfrak{g}}(T) \in \mathbf{Z}[\![T]\!]$. ♦

This proves the first part of Theorem 1.7. We note that Corollary 5.17 shows that we may not, in general, relax the assumptions in Corollary 8.16 and merely assume that $\mathfrak{g} \subset \mathfrak{gl}_d(\mathfrak{O})$ is a nilpotent subalgebra. The following completes the proof of Theorem 1.7.

Proposition 8.17. *Let* $\mathfrak{g} \subset \mathfrak{gl}_d(\mathfrak{O})$ *be an isolated subalgebra consisting of nilpotent matrices. Suppose that* $p \geqslant d$. *Then* $\mathsf{Z}^{\mathsf{cc}}_{\exp(\mathfrak{g})}(T) = \mathsf{Z}^{\mathsf{ask}}_{\mathrm{ad}(\mathfrak{g})}(T)$.

Proof. By Engel's theorem, $\mathfrak{g}$ is $\mathrm{GL}_d(K)$-conjugate to a subalgebra of $\mathfrak{n}_d(K)$ and hence nilpotent of class at most $d-1$; in particular, $\mathrm{ad}(a)^d = 0$ for all $a \in \mathfrak{g}$. By Corollary 8.15, $G := \exp(\mathfrak{g})$ is a saturable subgroup of $\mathrm{GL}_d(\mathfrak{O})$.

Let $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ denote the adjoint representation of G; hence, $\mathrm{Ad}(g)\colon \mathfrak{g} \to \mathfrak{g}$, $a \mapsto \log(\exp(a)^g) = a^g$ for $g \in G$. Recall that G_n denotes the image of G in $\mathrm{GL}_d(\mathfrak{O}_n)$ and $\mathfrak{g}_n$ that of $\mathfrak{g}$ in $\mathfrak{gl}_d(\mathfrak{O}_n)$. Clearly, $a \equiv 0 \pmod{\mathfrak{P}^n}$ if and only if $\exp(a) \equiv 1 \pmod{\mathfrak{P}^n}$ for $a \in \mathfrak{g}$. We may thus identify conjugacy classes of G_n with $\mathrm{Ad}(G)$-orbits on $\mathfrak{g}_n$. As $\mathfrak{g}$ is isolated within $\mathfrak{gl}_d(\mathfrak{O})$, we may identify $\mathfrak{g}_n = \mathfrak{g} \otimes \mathfrak{O}_n$ and obtain $\mathsf{Z}^{\mathsf{cc}}_{G}(T) = \mathsf{Z}^{\mathsf{oc}}_{\mathrm{Ad}(G)\curvearrowright\mathfrak{g}}(T)$.

By Corollary 8.15, $\mathrm{ad}(\mathfrak{g})$ is a saturable subalgebra of $\mathfrak{gl}(\mathfrak{g})$. The Hausdorff series shows that $\log(\exp(b)^{\exp(a)}) = \sum_{i=0}^{\infty} \frac{1}{i!}[b,_i a]$ for $a, b \in \mathfrak{g}$ (see [39, Eqn (3)]) whence $\mathrm{Ad}(\exp(a)) = \exp(\mathrm{ad}(a))$ for all $a \in \mathfrak{g}$. Thus, $\mathrm{Ad}(G) = \exp(\mathrm{ad}(\mathfrak{g}))$ and Corollary 8.16 shows that $\mathsf{Z}^{\mathsf{oc}}_{\mathrm{Ad}(G)\curvearrowright\mathfrak{g}}(T) = \mathsf{Z}^{\mathsf{ask}}_{\log(\mathrm{Ad}(G))\curvearrowright\mathfrak{g}}(T) = \mathsf{Z}^{\mathsf{ask}}_{\mathrm{ad}(\mathfrak{g})\curvearrowright\mathfrak{g}}(T)$. ♦

Remark 8.18. The conclusion of Proposition 8.17 does not generally hold if $\mathfrak{g}$ is not isolated. For a simple example, take $\mathfrak{g} = \mathfrak{P} \cdot \mathfrak{n}_2(\mathfrak{O})$. Then $\mathrm{ad}(\mathfrak{g}) = \{0\} \subset \mathfrak{gl}(\mathfrak{g}) \approx \mathfrak{gl}_1(\mathfrak{O})$ and thus $\mathsf{Z}^{\mathsf{ask}}_{\mathrm{ad}(\mathfrak{g})}(T) = 1/(1-qT) = 1 + qT + \mathcal{O}(T^2)$. On the other hand, the reduction of $\exp(\mathfrak{g})$ modulo $\mathfrak{P}$ is trivial whence $\mathsf{Z}^{\mathsf{cc}}_{\exp(\mathfrak{g})}(T) = 1 + T + \mathcal{O}(T^2)$.

Using the well-known equivalence between unipotent algebraic groups and nilpotent finite-dimensional Lie algebras over a field of characteristic zero (see [20, Ch. IV]), Corollary 8.16 and Proposition 8.17 now imply the following global result.

Corollary 8.19. *Let* k *be a number field with ring of integers* $\mathfrak{o}$. *Let* $\mathbf{G} \leqslant \mathrm{U}_d \otimes_{\mathbf{Z}} k$ *be a unipotent algebraic group over* k *and let* $\mathsf{G} \leqslant \mathrm{U}_d \otimes_{\mathbf{Z}} \mathfrak{o}$ *be the associated* $\mathfrak{o}$*-form of* $\mathbf{G}$ *(i.e. the schematic closure of* $\mathbf{G}$*). Let* $\boldsymbol{\mathfrak{g}} \subset \mathfrak{n}_d(k)$ *be the Lie algebra of* $\mathbf{G}$ *and* $\mathfrak{g} = \boldsymbol{\mathfrak{g}} \cap \mathfrak{n}_d(\mathfrak{o})$. *Then for almost all* $v \in \mathcal{V}_k$, $\mathsf{Z}^{\mathsf{oc}}_{\mathsf{G}(\mathfrak{o}_v)}(T) = \mathsf{Z}^{\mathsf{ask}}_{\mathfrak{g}_v}(T)$ *and* $\mathsf{Z}^{\mathsf{cc}}_{\mathsf{G}(\mathfrak{o}_v)}(T) = \mathsf{Z}^{\mathsf{ask}}_{\mathrm{ad}(\mathfrak{g}_v)}(T)$. ♦

Using Theorem 4.18, we further establish the following functional equations for orbit-counting and conjugacy class zeta functions arising from unipotent algebraic groups.

Corollary 8.20. *Let the notation be as in Corollary 8.19. Then for almost all* $v \in \mathcal{V}_k$,

$$\mathsf{Z}^{\mathsf{oc}}_{\mathsf{G}(\mathfrak{o}_v)}(T) \Bigg|_{(q_v,T)\to(q_v^{-1},T^{-1})} = (-q_v^d T) \cdot \mathsf{Z}^{\mathsf{oc}}_{\mathsf{G}(\mathfrak{o}_v)}(T)$$

and

$$\mathsf{Z}^{\mathsf{cc}}_{\mathsf{G}(\mathfrak{o}_v)}(T)\,\Big|_{(q_v,T)\to(q_v^{-1},T^{-1})} = (-q_v^{\dim_k(\mathbf{G})}T)\cdot \mathsf{Z}^{\mathsf{cc}}_{\mathsf{G}(\mathfrak{o}_v)}(T). \qquad \blacklozenge$$

Corollary 8.21. *Let k be a number field with ring of integers $\mathfrak{o}$. Let $\mathbf{G} \leqslant \mathrm{U}_d \otimes_{\mathbf{Z}} k$ and $\mathbf{H} \leqslant \mathrm{U}_e \otimes_{\mathbf{Z}} k$ be unipotent algebraic groups over k with $\mathfrak{o}$-forms G and H as above. Suppose that for almost all $v \in \mathcal{V}_k$, $\mathsf{Z}^{\mathsf{cc}}_{\mathsf{G}(\mathfrak{o}_v)}(T) = \mathsf{Z}^{\mathsf{cc}}_{\mathsf{H}(\mathfrak{o}_v)}(T)$. Then $\dim_k(\mathbf{G}) = \dim_k(\mathbf{H})$.* $\blacklozenge$

Proof. Corollary 8.20 and [77, §4] allow us to recover $\dim_k(\mathbf{G})$ from $\big(\mathsf{Z}^{\mathsf{cc}}_{\mathsf{G}(\mathfrak{o}_v)}(T)\big)_{v\in\mathcal{V}_k\setminus S}$ for any finite set $S \subset \mathcal{V}_k$. $\blacklozenge$

We note that there are examples of non-isomorphic groups $\mathbf{G}$ and $\mathbf{H}$ (of the same dimension) which satisfy $\mathsf{Z}^{\mathsf{cc}}_{\mathsf{G}(\mathfrak{o}_v)}(T) = \mathsf{Z}^{\mathsf{cc}}_{\mathsf{H}(\mathfrak{o}_v)}(T)$ for almost all $v \in \mathcal{V}_k$; see Table 2 in §9.3.

9 Further examples

9.1 Computer calculations: Zeta

The author's software package Zeta [78] for Sage [84] can compute numerous types of "generic local" zeta functions in fortunate ("non-degenerate") cases. The techniques used by Zeta were developed over the course of several papers; see [79], in particular, and [80] for an overview and references to other pieces of software that Zeta relies upon. When performing computations, Zeta proceeds by attempting to explicitly compute certain types of p-adic integrals. Fortunately, the integrals in (4.3) and (4.6) can both be encoded in terms of the "representation data" introduced in [75, §5] whence the author's computational techniques apply verbatim to the functions $\mathsf{Z}_{M\otimes_{\mathbf{Z}}\mathfrak{O}}(T)$, where M is $\mathbf{Z}$-defined. In detail, given a submodule $M \subset \mathrm{M}_{d\times e}(\mathbf{Z})$, Zeta can be used to attempt to construct a rational function $W(X,T) \in \mathbf{Q}(X,T)$ with the following property: for almost all primes p and all finite extensions $K/\mathbf{Q}_p$, $\mathsf{Z}_{M\otimes_{\mathbf{Z}}\mathfrak{O}_K}(T) = W(q_K,T)$; we note that for various reasons, Zeta may well fail to construct $W(X,T)$ even if it exists. Given $M \subset \mathrm{M}_{d\times e}(\mathbf{Z})$, Zeta can also be used to attempt to construct a formula as in Theorem 4.11. We note that while the techniques used by Zeta can, at least in principle, be used to construct an explicit number C_M such that all primes p which needed to be excluded above satisfy $p < C_M$, such a number C_M is not presently determined.

The remainder of this section is devoted to a number of examples of functions $\mathsf{Z}_M(T)$ and $\mathsf{Z}^{\mathsf{cc}}_G(T)$ (via Theorem 1.7) computed with the help of Zeta. Throughout, $\mathfrak{O}$ denotes the valuation ring of a non-Archimedean local field $K \supset \mathbf{Q}_p$ with residue field size q.

9.2 Examples of ask zeta functions

Example 9.1 (Small poles and unbounded denominators)**.** Let

$$M = \left\{ \begin{bmatrix} a & b & a \\ b & c & d \\ a & d & c \end{bmatrix} : a,b,c,d \in \mathfrak{O} \right\}.$$

Then for sufficiently large p,

$$\mathsf{Z}_M(T) = \frac{1 + 5q^{-1}T - 12q^{-2}T + 5q^{-3}T + q^{-4}T^2}{(1 - q^{-1}T)(1 - T)^2}.$$

Hence, the real poles of $\zeta_M(s)$ are -1 and 0; it is easy to see that $\operatorname{gor}(M) = 3$ (see Definition 3.2). This example illustrates that, in contrast to the case of $\mathrm{M}_{d\times e}(\mathfrak{O})$, $d - \operatorname{gor}(M)$ is generally not a lower bound for the real poles of $\zeta_M(s)$. Note that $\mathsf{Z}_M(T)$ has unbounded denominators—the author has found comparatively few modules of square matrices with this property (and initially suspected they did not exist).

Example 9.2. Suppose that $p \neq 2$ and let

$$M = \left\{ \begin{bmatrix} 0 & x_2 & -x_3 & 0 & 0 & x_1 \\ 0 & 0 & x_1 & \frac{x_2}{2} & -\frac{x_3}{2} & x_5 \\ 0 & 0 & 0 & x_1 & 0 & x_4 \\ 0 & 0 & 0 & 0 & x_1 & x_3 \\ 0 & 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} : x_1, \dots, x_5 \in \mathfrak{O} \right\}.$$

For sufficiently large p,

$$\begin{aligned}
\mathsf{Z}_M(T) = \big((&+q^{36}T^{19} - 4q^{35}T^{19} - q^{34}T^{20} + q^{35}T^{18} + 8q^{34}T^{19} - 2q^{34}T^{18} - 2q^{33}T^{19} - q^{34}T^{17} \\
&- 6q^{33}T^{18} - q^{32}T^{19} + 3q^{33}T^{17} + 5q^{32}T^{18} + 3q^{32}T^{17} + 6q^{31}T^{18} - q^{32}T^{16} - 12q^{31}T^{17} \\
&- 2q^{30}T^{18} - 9q^{30}T^{17} - q^{31}T^{15} + 14q^{30}T^{16} + 14q^{29}T^{17} + 4q^{30}T^{15} + 5q^{29}T^{16} \\
&- 3q^{28}T^{17} - 14q^{29}T^{15} - 41q^{28}T^{16} + q^{29}T^{14} + 12q^{28}T^{15} + 26q^{27}T^{16} - 2q^{28}T^{14} \\
&+ 46q^{27}T^{15} - 4q^{26}T^{16} - 7q^{27}T^{14} - 73q^{26}T^{15} + 2q^{27}T^{13} - 24q^{26}T^{14} + 32q^{25}T^{15} \\
&- 2q^{26}T^{13} + 103q^{25}T^{14} - 3q^{24}T^{15} + 6q^{25}T^{13} - 98q^24T^{14} - q^25T^12 - 89q^24T^{13} \\
&+ 29q^{23}T^{14} + 8q^{24}T^{12} + 176q^{23}T^13 - 2q^{22}T^14 + 35q^{23}T^{12} - 115q^{22}T^{13} + q^{23}T^{11} \\
&- 178q^{22}T^{12} + 25q^{21}T^{13} - 15q^{22}T^{11} + 223q^{21}T^{12} - 2q^{20}T^{13} + 119q^{21}T^{11} \\
&- 100q^{20}T^{12} + q^{21}T^{10} - 262q^{20}T^{11} + 16q^{19}T^{12} - 39q^{20}T^{10} + 214q^{19}T^{11} - q^{18}T^{12} \\
&+ 176q^{19}T^{10} - 61q^{18}T^{11} + 3q^{19}T^{9} - 280q^{18}T^{10} + 3q^{17}T^{11} - 61q^{18}T^{9} + 176q^{17}T^{10} \\
&- q^{18}T^{8} + 214q^{17}T^{9} - 39q^{16}T^{10} + 16q^{17}T^{8} - 262q^{16}T^{9} + q^{15}T^{10} - 100q^{16}T^{8} \\
&+ 119q^{15}T^{9} - 2q^{16}T^{7} + 223q^{15}T^{8} - 15q^{14}T^{9} + 25q^{15}T^{7} - 178q^{14}T^{8} + q^{13}T^{9} \\
&- 115q^{14}T^{7} + 35q^{13}T^{8} - 2q^{14}T^{6} + 176q^{13}T^{7} + 8q^{12}T^{8} + 29q^{13}T^{6} - 89q^{12}T^{7} \\
&- q^{11}T^{8} - 98q^{12}T^{6} + 6q^{11}T^{7} - 3q^{12}T^{5} + 103q^{11}T^{6} - 2q^{10}T^{7} + 32q^{11}T^{5} - 24q^{10}T^{6} \\
&+ 2q^{9}T^{7} - 73q^{10}T^{5} - 7q^{9}T^{6} - 4q^{10}T^{4} + 46q^{9}T^{5} - 2q^{8}T^{6} + 26q^{9}T^{4} + 12q^{8}T^{5} + q^{7}T^{6} \\
&- 41q^{8}T^{4} - 14q^{7}T^{5} - 3q^{8}T^{3} + 5q^{7}T^{4} + 4q^{6}T^{5} + 14q^{7}T^{3} + 14q^{6}T^{4} - q^{5}T^{5} \\
&- 9q^{6}T^{3} - 2q^{6}T^{2} - 12q^{5}T^{3} - q^{4}T^{4} + 6q^{5}T^{2} + 3q^{4}T^{3} + 5q^{4}T^{2} + 3q^{3}T^{3} - q^{4}T \\
&- 6q^{3}T^{2} - q^{2}T^{3} - 2q^{3}T - 2q^{2}T^{2} + 8q^{2}T + qT^{2} - q^{2} - 4qT + T\big) \\
&/\big(q^{2}(1 - q^{10}T^{5})(1 - q^{8}T^{4})(1 - q^{5}T^{3})(1 - q^{4}T^{2})^{2}(1 - q^{3}T^{2})(1 - q^{2}T)(1 - qT)^{2}\big).
\end{aligned}$$

Since $\mathsf{Z}_M(T) = 1 + (2q^2 + 4q + 4q^{-1} - q^{-2} - 8)T + \mathcal{O}(T^2)$, we see that, in contrast to O-maximal or K-minimal cases (see §5–6), the complexity of $\mathrm{ask}(M_1 \mid V_1)$ is in general a poor indicator of that of $\mathsf{Z}_M(T)$.

We note that by Corollary 8.16 and since $\mathsf{Z}_M(T) \notin \mathbf{Z}[\![T]\!]$, the module M cannot be a Lie subalgebra of $\mathfrak{n}_6(\mathfrak{O})$. Indeed, this is readily verified directly even though M is listed among Lie algebras in [73, Table 5].

Rojas's article [73] provides numerous examples of Lie subalgebras $\mathfrak{g} \subset \mathfrak{n}_d(\mathbf{Z})$, say, for $d \leqslant 6$. For many of these, we may use Zeta to compute $\mathsf{Z}_{\mathfrak{g}\otimes_{\mathbf{Z}}\mathbf{Z}_p}(T)$. Here, we only include one example.

Example 9.3. Let $p \neq 2$ and let

$$\mathfrak{g} = \left\{ \begin{bmatrix} 0 & x_1 & \frac{x_2}{2} & -\frac{x_3}{2} & x_5 \\ 0 & 0 & x_1 & 0 & x_4 \\ 0 & 0 & 0 & x_1 & x_3 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} : x_1, \dots, x_5 \in \mathfrak{O} \right\}.$$

Then $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{n}_5(\mathfrak{O})$ of nilpotency class 4, listed as $L_{5,6}$ (de Graaf's [19] notation) in [73, Table 3]. For sufficiently large p,

$$\begin{aligned} \mathsf{Z}_{\mathfrak{g}}(T) = (&+ q^8T^7 - 3q^8T^6 + q^8T^5 + q^7T^6 + 2q^7T^5 - 2q^6T^5 - 2q^6T^4 - q^5T^5 + 6q^5T^4 \\ &- 3q^4T^4 - 3q^4T^3 + 6q^3T^3 - q^3T^2 - 2q^2T^3 - 2q^2T^2 + 2qT^2 + qT + T^2 \\ &- 3T + 1)/((1 - q^5T^3)(1 - q^4T^2)(1 - q^2T)(1 - qT)^2). \end{aligned}$$

Numerous examples (including the case of $\mathrm{M}_{d\times e}(\mathfrak{O})$) show that $\zeta_M(s)$ may have a pole at zero and Example 9.1 shows that negative poles can arise even for modules of square matrices. In contrast, all of the author's computations are consistent with the following question having a positive answer.

Question 9.4. Let k be a number field with ring of integers $\mathfrak{o}$. Let $\mathfrak{g} \subset \mathfrak{n}_d(\mathfrak{o})$ be a Lie subalgebra. Is it the case that for almost all $v \in \mathcal{V}_k$, every real pole of $\zeta_{\mathfrak{g}_v}(s)$ is positive?

Supposing that Question 9.4 indeed has a positive answer, if $\mathbf{G} \leqslant \mathrm{U}_d \otimes_{\mathbf{Z}} k$ is an algebraic group over k with associated $\mathfrak{o}$-form $\mathsf{G} \leqslant \mathrm{U}_d \otimes_{\mathbf{Z}} \mathfrak{o}$ (see Corollary 8.19), then we may evaluate the meromorphic continuation of $\mathsf{Z}^{\mathsf{oc}}_{\mathsf{G}(\mathfrak{o}_v)}(q_v^{-s})$ at $s = 0$ for almost all $v \in \mathcal{V}_k$. Inspired by similar questions regarding the behaviour at zero of local subalgebra [74, Conj. IV], submodule [76, Conj. E], and representation [79, Qu. 8.5] zeta functions, it would then be interesting to see if one can interpret the resulting rational numbers, say in terms of properties of the orbit space $\mathfrak{o}_v^d/\mathsf{G}(\mathfrak{o}_v)$.

9.3 Examples of conjugacy class zeta functions

Let k be a number field with ring of integers $\mathfrak{o}$. Morozov [70] classified nilpotent Lie algebras of dimension at most 6 over an arbitrary field of characteristic zero—equivalently, he classified unipotent algebraic groups of dimension at most 6 over these fields. A recent computer-assisted version of this classification (valid for fields of characteristic $\neq 2$) is due to de Graaf [19]. We use his notation and let $L_{d,i}$ (or $L_{d,i}(a)$) denote the ith Lie k-algebra (with parameter a) given in [19, §4].

Table 1 provides a complete list of "generic conjugacy class zeta function" associated with nilpotent Lie k-algebras of dimension at most 5 in the following sense: for each such algebra $\mathfrak{g}$, let $\mathbf{G}$ be its associated unipotent algebraic group over k. After choosing an embedding $\mathbf{G} \leqslant \mathrm{U}_d \otimes_{\mathbf{Z}} k$, we obtain an $\mathfrak{o}$-form G of $\mathbf{G}$ as in Corollary 8.19. Then for almost all $v \in \mathcal{V}_k$ and all finite extensions K/k_v, $\mathsf{Z}^{\mathsf{cc}}_{\mathsf{G}(\mathfrak{O})}(T)$ is given in Table 1.

In contrast to dimension at most 5, Zeta is unable to compute generic conjugacy class zeta functions associated with every nilpotent Lie k-algebra of dimension 6. Nevertheless, Table 2 contains numerous examples of such functions; we only included examples corresponding to $\oplus$-indecomposable algebras. Clearly, generic conjugacy class zeta functions of direct products of algebraic groups are Hadamard products of the zeta functions corresponding to the factors. We note that $L_{3,2} \approx \mathfrak{n}_3(K)$ and $L_{6,19}(-1) \approx \mathfrak{n}_4(K)$. A formula for $\mathsf{Z}^{\mathsf{cc}}_{\mathrm{U}_3(\mathfrak{O})}(T)$ was previously given in [6, §8.2]. This formula is incorrect due to a sign mistake. More substantially, the computation in [6, §8.2] relies on [6, Prop. 6.2] and what seems to be a variation of the integral formalism developed in [6] for unipotent groups; this is however not explained. Said integral formalism in [6] appears to be essentially different from the methods developed and applied here.

Possible further directions. A refinement of Higman's conjecture (see §1) predicts that $\mathrm{k}(\mathrm{U}_d(\mathbf{F}_q))$ is a polynomial in $q-1$ with non-negative coefficients. In recent years, the same question has been studied for groups of $\mathbf{F}_q$-rational points of unipotent radicals of Borel subgroups of more general algebraic groups such as Chevalley groups of small rank; see, in particular, work of Goodwin et al. [40–43]. In this spirit, an elementary calculation using the formulae in Table 1–2 shows that the coefficients of the generic conjugacy class zeta functions associated with $\mathfrak{n}_3(K)$ and $\mathfrak{n}_4(K)$ are polynomials with non-negative coefficients in $q-1$, generalising the known cases of the corresponding coefficients of T. The same is true of the generic conjugacy class zeta functions associated with $L_{4,3}$; the latter algebra is isomorphic to the nilradical of a Borel subalgebra of $\mathfrak{sp}_4(K)$. It would be interesting to further explore to what extent such non-negativity properties are satisfied by the coefficients of ask zeta functions in the setting of Goodwin et al.

For another intriguing problem, let $\mathfrak{f}_{c,d}$ be the free nilpotent Lie ring of class c on d generators and write $\mathfrak{f}_{c,d}(R) := \mathfrak{f}_{c,d} \otimes_{\mathbf{Z}} R$. O'Brien and Voll [71, §5] gave a combinatorial description of $\mathrm{k}(\exp(\mathfrak{f}_{c,d}(\mathbf{F}_q)))$ under mild assumptions on q. The generic conjugacy class zeta functions associated with $\mathfrak{f}_{3,2}$ and $\mathfrak{f}_{2,3}$ can be found in Tables 1 and 2, respectively. Lins computed the conjugacy class zeta functions associated with $\mathfrak{f}_{2,d}$ for all d; see [68, Cor. 1.5]. It seems challenging to determine the conjugacy class zeta functions $\mathfrak{f}_{c,d}$ in general.

$\mathfrak{g}$	$\mathsf{Z}^{\mathsf{cc}}_{\mathsf{G}(\mathfrak{O}))}(T)$
$L_{1,1}$	$1/(1-qT)$
$L_{2,1}$	$1/(1-q^2T)$
$L_{3,1}$	$1/(1-q^3T)$
$L_{3,2} \approx \mathfrak{n}_3(K)$	$(1-T)/((1-q^2T)(1-qT))$
$L_{4,1}$	$1/(1-q^4T)$
$L_{4,2}$	$(1-qT)/((1-q^3T)(1-q^2T))$
$L_{4,3}$	$(1-T)/(1-q^2T)^2$
$L_{5,1}$	$1/(1-q^5T)$
$L_{5,2}$	$(1-q^2T)/((1-q^4T)(1-q^3T))$
$L_{5,3}$	$(1-qT)/(1-q^3T)^2$
$L_{5,4}$	$(1-T)/((1-q^4T)(1-qT))$
$L_{5,5}$	$\frac{1-T-qT+q^2T+q^2T^2-q^3T^2-q^4T^2+q^4T^3}{(1-q^5T^2)(1-q^3T)(1-qT)}$
$L_{5,6}$	$\frac{1-2T+qT^2+q^2T-2q^3T^2+q^3T^3}{(1-q^5T^2)(1-q^2T)(1-qT)}$
$L_{5,7}$	$(1-T)/((1-q^3T)(1-q^2T))$
$L_{5,8}$	$(1-qT)/(1-q^3T)^2$
$L_{5,9} \approx \mathfrak{f}_{3,2}(K)$	$(1-T)/((1-q^3T)(1-q^2T))$

Table 1: Complete list of generic conjugacy class zeta functions of unipotent algebraic groups of dimension at most 5

$\mathfrak{g}$	$\mathsf{Z}^{\mathsf{cc}}_{\mathsf{G}(\mathfrak{O})}(T)$
$L_{6,10},\ L_{6,25};\ L_{6,26} \approx \mathfrak{f}_{2,3}(K)$	$(1-qT)/((1-q^4T)(1-q^3T))$
$L_{6,11},\ L_{6,12},\ L_{6,20}$	$\frac{1-2qT+q^2T+q^4T^2-2q^5T^2+q^6T^3}{(1-q^6T^2)(1-q^3T)^2}$
$L_{6,16}$	$(1-qT)(1-T)/((1-q^2T)^2(1-q^3T))$
$L_{6,17}$	$\frac{1-T-qT+q^2T+q^3T^2-q^4T^2-q^5T^2+q^5T^3}{(1-q^6T^2)(1-q^3T)(1-q^2T)}$
$L_{6,18}$	$(1-T)/((1-q^2T)(1-q^4T))$
$L_{6,19}(0)$	$\frac{1+T-3qT-q^2T+q^3T^2+3q^4T^2-q^5T^2-q^5T^3}{(1-q^3T)^3(1-q^2T)}$
$L_{6,19}(-1) \approx \mathfrak{n}_4(K),\ L_{6,21}(0)$	$(1-qT)^2/((1-q^3T)^2(1-q^2T))$
$L_{6,21}(1)$	$\frac{1-T-qT+q^2T+q^2T^2-q^3T^2-q^4T^2+q^4T^3}{(1-q^5T^2)(1-q^3T)(1-q^2T)}$
$L_{6,22}(0)$	$\frac{1-qT-q^2T+q^3T+q^4T^2-q^5T^2-q^6T^2+q^7T^3}{(1-q^7T^2)(1-q^4T)(1-q^2T)}$
$L_{6,23},\ L_{6,24}(0)$	$\frac{1-2qT+q^3T+q^3T^2-2q^5T^2+q^6T^3}{(1-q^7T^2)(1-q^3T)(1-q^2T)}$

Table 2: Examples of generic conjugacy class zeta functions of unipotent algebraic groups of dimension 6

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