Alternating Sign Matrices and Latin Squares

Cian O’Brien
Rachel Quinlan and Kevin Jennings

National University of Ireland, Galway

c.obrien40@nuigalway.ie

Postgraduate Modelling Group, NUI Galway
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An *Alternating Sign Matrix (ASM)* is \((0, 1, -1)\)-matrix for which all row and column sums are 1, and the non-zero elements in each row and column alternate in sign.
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\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & -1 & 1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]
An *Alternating Sign Matrix (ASM)* is \((0, 1, -1)\)-matrix for which all row and column sums are 1, and the non-zero elements in each row and column alternate in sign.

\[
\begin{bmatrix}
0 & + & 0 & 0 \\
+ & - & + & 0 \\
0 & + & - & + \\
0 & 0 & + & 0
\end{bmatrix}
\]
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They are a generalisation of the permutation matrices.
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\begin{bmatrix}
+ & 0 & 0 & 0 \\
0 & 0 & 0 & + \\
0 & + & 0 & 0 \\
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\end{bmatrix}
\]
An *Alternating Sign Hypermatrix (ASHM)* is a hypermatrix $(n \times n \times n$ array) for which every plane is an ASM.
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\[
\begin{array}{ccc}
+ & 0 & 0 \\
0 & + & 0 \\
0 & 0 & + \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & + & 0 \\
+ & 0 & 0 \\
0 & 0 & + \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & 0 & + \\
+ & 0 & 0 \\
0 & + & 0 \\
\end{array}
\]
The number of non-zero entries in the rows/columns of an ASM is bounded above by \((1, 3, 5, \ldots, 5, 3, 1)\).
Non-Zero Entry Bounds for ASMs

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\[
\begin{bmatrix}
0 & 0 & + & 0 & 0 \\
0 & + & - & + & 0 \\
+ & - & + & - & + \\
0 & + & - & + & 0 \\
0 & 0 & + & 0 & 0 \\
\end{bmatrix}
\]
The number of non-zero entries in the planes of an ASHM is bounded above by

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
1 & 3 & \cdots & 3 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 3 & \cdots & 3 & 1 \\
1 & 1 & \cdots & 1 & 1 \\
\end{bmatrix}
\]
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\begin{pmatrix}
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1 & 3 & \cdots & 3 & 1 \\
1 & 1 & \cdots & 1 & 1
\end{pmatrix}
\]
Latin Squares

An $n \times n$ Latin Square is an $n \times n$ array of $n$ symbols such that each symbol occurs exactly once in each row and column.
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$$
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{bmatrix}
$$
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$$\begin{bmatrix}
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\end{bmatrix}$$

Each $n \times n$ latin square can be decomposed uniquely into a sum of scalar multiples of mutually orthogonal permutation matrices.
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\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix}
+ 2 \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
+ 3 \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}
\]

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\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix} + 2 \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} + 3 \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}
\]

Each $n \times n$ latin square can be decomposed uniquely into a sum of scalar multiples of mutually orthogonal permutation matrices. Therefore each latin square corresponds uniquely to a permutation hypermatrix. For a permutation hypermatrix $M$, define $L(M)$ to be $L(M)_{i,j,k} = \sum_{k=1}^{n} k \times M_{i,j,k}$.
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$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 3 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} + & 0 & 0 \\ 0 & 0 & + \\ 0 & + & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & + & 0 \\ + & 0 & 0 \\ 0 & 0 & + \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & + \\ 0 & + & 0 \\ + & 0 & 0 \end{bmatrix}$$
An $n \times n$ ASHM-Latin Square is an $n \times n$ matrix $L(A)$ such that $L(A)_{i,j,k} = \sum_{k=1}^{n} k \times A_{i,j,k}$ for some $n \times n \times n$ alternating sign hypermatrix $A$. 
**ASHM-Latin Squares**

An $n \times n$ **ASHM-Latin Square** is an $n \times n$ matrix $L(A)$ such that

$$L(A)_{i,j,k} = \sum_{k=1}^{n} k \times A_{i,j,k}$$

for some $n \times n \times n$ alternating sign hypermatrix $A$

$$A = \begin{bmatrix}
+ & 0 & 0 \\
0 & + & 0 \\
0 & 0 & +
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & + & 0 \\
+ & - & + \\
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\end{bmatrix} \rightarrow \begin{bmatrix}
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$$A = \begin{bmatrix} + & 0 & 0 \\ 0 & + & 0 \\ 0 & 0 & + \end{bmatrix} \rightarrow \begin{bmatrix} 0 & + & 0 \\ + & - & + \\ 0 & + & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & + \\ 0 & + & 0 \\ + & 0 & 0 \end{bmatrix}$$

$$L(A) = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$
All entries of an $n \times n$ ASHM-Latin Square are between 1 and $n$. 
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The outer planes of the ASHM can each only contain one entry.
Basic Facts about ASHM-Latin Squares

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Each row and column sum of the ASHM is 1, and row and column of
and ASHM-Latin Square is therefore $1(1) + 2(1) + 3(1) + \cdots + n(1)$. 
Interesting Open Problems/Questions

Does each ASHM-Latin Square correspond to exactly one ASHM?
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What is the maximum number of times a symbol can occur in a ASHM-Latin Square?

Current highest found is $2n$, with achievable example for all odd $n \geq 5$

$$\begin{bmatrix}
1 & 2 & 3 & 7 & 5 & 6 & 4 \\
2 & 2 & 3 & 3 & 6 & 7 & 5 \\
3 & 3 & 2 & 5 & 3 & 5 & 7 \\
4 & 3 & 4 & 2 & 6 & 3 & 6 \\
6 & 7 & 3 & 4 & 2 & 3 & 3 \\
7 & 5 & 6 & 3 & 3 & 2 & 2 \\
5 & 6 & 7 & 4 & 3 & 2 & 1
\end{bmatrix}$$