Alternating Sign Matrices and Latin Squares

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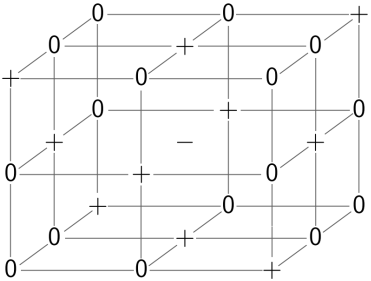

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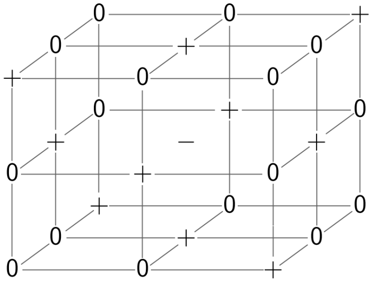

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Each row and column sum of the ASHM is 1, and row and column of and ASHM-Latin Square is therefore $1(1) + 2(1) + 3(1) + \cdots + n(1)$.

Interesting Open Problems/Questions

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Current highest found is $2n$, with achievable example for all odd $n \geq 5$

$$\begin{bmatrix} 1 & 2 & 3 & 7 & 5 & 6 & 4 \\ 2 & 2 & 3 & 3 & 6 & 7 & 5 \\ 3 & 3 & 2 & 5 & 3 & 5 & 7 \\ 4 & 3 & 4 & 2 & 6 & 3 & 6 \\ 6 & 7 & 3 & 4 & 2 & 3 & 3 \\ 7 & 5 & 6 & 3 & 3 & 2 & 2 \\ 5 & 6 & 7 & 4 & 3 & 2 & 1 \end{bmatrix}$$

Richard A. Brualdi, Geir Dahl, *Alternating Sign Matrices and Hypermatrices, and a Generalization of Latin Squares*. arXiv:1704.07752, 2017.