Master Stability Function: an introduction

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Lyapunov Exponents

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$$|\delta\mathbf{Z}(t)| \approx e^{\lambda t} \left|\delta\mathbf{Z}_0\right|, \tag{3}$$

where λ is the leading Lyapunov exponent.

Computing Lyapunov Exponents

Lyapunov Exponents are defined as

$$\lambda = \lim_{t\to\infty} \lim_{\delta \mathbf{Z}_0 \to 0} \frac{1}{t} \ln \frac{\|\delta \mathbf{Z}(t)\|}{\|\delta \mathbf{Z}_0(t),\|} \tag{4}$$

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When one only has access to experimental data it is usually impossible to calculate Lyapunov Exponents.

Consider the following dynamical systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{6}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f}$ is a nonlinear vector field.

Coupling N such systems (agents or nodes), according to the topology of any graph G, leads to the following set of $N \times n$ ordinary differential equations

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) - \sigma \sum_{j=1}^{N} L_{ij}\mathbf{h}(\mathbf{x}_j), \quad i = 1, \dots, N, \tag{7}$$

where $\mathbf{h}$ is a coupling function and L_{ij} is the ij-th entry of the Laplacian matrix $\mathbf{L}$.

Laplacian Matrix

The Laplacian matrix is given by $\mathbf{L} = \mathbf{D} - \mathbf{A}$, where $\mathbf{D}$ is a diagonal matrix whose diagonal entries are the number of connections of each node (the degree d_i of node i) and $\mathbf{A}$ is the adjacency matrix of G .

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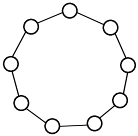

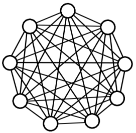

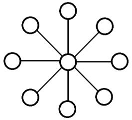

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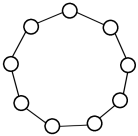

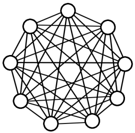

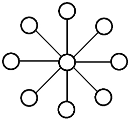

$$\begin{bmatrix} 2 & -1 & \cdots & -1 \\ -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 2 \end{bmatrix} \begin{bmatrix} N-1 & -1 & \cdots & -1 \\ -1 & N-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & N-1 \end{bmatrix} \begin{bmatrix} N-1 & -1 & \cdots & -1 \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 1 \end{bmatrix} \tag{9}$$

Synchronization manifolds

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An central question is: for which values of coupling strength σ is the synchronization manifold **s** stable?

Linearization around the synchronization manifold

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consider the variational equations about the synchronization manifold **s**,

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where $\mathbf{J}_{\mathbf{f}}(\mathbf{x})$ and $\mathbf{J}_{\mathbf{h}}(\mathbf{x})$ are the Jacobians of **f** and **h**, respectively.

Block-Diagonally Decomposition

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can be block-diagonally decoupled as

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Normalized Variational Equations

Let us focus on the last equation

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- All the other eigenvalues correspond to the directions transverse to the synchronization manifold.

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The Master Stability Function is the largest Lyapunov exponent $\lambda_{\max}$ for the generic variational equation

$$\delta\dot{\mathbf{y}}_i = \left(\mathbf{J_f}(\mathbf{x}, \mathbf{p})|_{\mathbf{x}=\mathbf{s}} + (\alpha + i\beta)\, \mathbf{J_h}(\mathbf{x})|_{\mathbf{x}=\mathbf{s}} \right) \delta\mathbf{y}_i, \quad i = 2, \ldots, N. \tag{20}$$

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If all the eigenmodes are stable, then the synchronization manifold is stable for that value of σ.

Thank you

Pecora, Louis M., and Thomas L. Carroll. ”Master stability functions for synchronized coupled systems.” Physical review letters 80.10 (1998): 2109.