Almost-nonsingular Entry Pattern Matrices

Hieu Ha Van
Supervised by: Dr. Rachel Quinlan
School of Maths, Statistics and Applied Maths

NUI Galway

February 14, 2019

NUI Galway

Entry pattern matrices

An entry pattern matrix (EPM for short) is a matrix in which:

- Each entry is an element of a specified set of independent indeterminates.
- Entries can be the same, but can not be a constant.

Entry pattern matrices

An entry pattern matrix (EPM for short) is a matrix in which:

- Each entry is an element of a specified set of independent indeterminates.
- Entries can be the same, but can not be a constant.

Example

Let

$$A(x,y,z) = \begin{bmatrix} x & y \\ z & x \end{bmatrix}, B = \begin{bmatrix} x+y & 0 \\ z & x \end{bmatrix}$$

Then A is an entry pattern matrix with 3 indeterminates $\{x,y,z\}$ while B is not.

Almost-nonsingular EPMs

An $\mathbb{F}$-completion of A is a matrix obtained by assigning elements of $\mathbb{F}$ to indeterminates appearing in A.

Almost-nonsingular EPMs

An $\mathbb{F}$-completion of A is a matrix obtained by assigning elements of $\mathbb{F}$ to indeterminates appearing in A.

A square EPM $A(x_1, \cdots, x_k)$ is said to be almost-nonsingular over a field $\mathbb{F}$ (or $\mathbb{F}$-almost-nonsingular) if

$$\det A(a_1, \cdots, a_k) = 0 \Leftrightarrow a_1 = a_2 = \cdots = a_k.$$

Almost-nonsingular EPMs

An $\mathbb{F}$-completion of A is a matrix obtained by assigning elements of $\mathbb{F}$ to indeterminates appearing in A.

A square EPM $A(x_1, \cdots, x_k)$ is said to be almost-nonsingular over a field $\mathbb{F}$ (or $\mathbb{F}$-almost-nonsingular) if

$$\det A(a_1, \cdots, a_k) = 0 \Leftrightarrow a_1 = a_2 = \cdots = a_k.$$

Example

Let

$$A(x,y) = \begin{bmatrix} x & y & x & x \\ y & y & x & y \\ x & x & x & y \\ x & y & y & y \end{bmatrix}.$$

Then $\det A(a,b) = (a-b)^4$ over any field. Hence, A is almost-nonsingular over any field.

Almost-nonsingular EPMs

Let $\tau_{\mathbb{F}}(n)$ be the **maximum possible number** of indeterminates in an $n \times n$ EPM that is almost-nonsingular over $\mathbb{F}$. In the previous talk, I have given the bounds for $\tau_{\mathbb{F}}(n)$ for any field $\mathbb{F}$ and any integer n and in particular, the exactly value of $\tau_{\mathbb{R}}(n)$ if n has an odd divisor greater than 3. This talk is to

- construct $\mathbb{Q}$-almost-nonsingular EPMs from $\mathbb{Q}$-irreducible polynomials. And
- prove that $\tau_{\mathbb{Q}}$ is bounded below by an increasing linear function on $\mathbb{N}$.

Universally Almost-nonsingular EPMs

Theorem

An EPM A is almost-nonsingular over any field only if it has two indeterminates. And for every $n \geq 4$, there exists such an almost-nonsingular EPM of size $n \times n$.

$$T_4 = A$$

$$T_{n+1} = \begin{cases} \begin{bmatrix} T_n & c_n \\ c_n^T & x \end{bmatrix} & \text{if } n \text{ is even} \\ \begin{bmatrix} T_n & c_n \\ c_n^T & y \end{bmatrix} & \text{if } n \text{ is odd} \end{cases}$$

where c_n is the last column of T_n.

Universally Almost-nonsingular EPMs

Theorem

An EPM A is almost-nonsingular over any field only if it has two indeterminates. And for every $n \geq 4$, there exists such an almost-nonsingular EPM of size $n \times n$.

$$T_4 = A$$

$$T_{n+1} = \begin{cases} \begin{bmatrix} T_n & c_n \\ c_n^T & x \end{bmatrix} & \text{if } n \text{ is even} \\ \begin{bmatrix} T_n & c_n \\ c_n^T & y \end{bmatrix} & \text{if } n \text{ is odd} \end{cases}$$

where c_n is the last column of T_n. Since

$$\det T_{n+1} = (x - y) \det T_n$$

Nonsingular Pseudo-EPMs

Let $S = \{x_1, x_2, \cdots, x_k\}$ be a set of independent indeterminates. We say that a matrix A whose entries are either zero or $\pm x_i$ is a **pseudo-EPM.**

Example

Let

$$A = \begin{bmatrix} x & y \\ z & x \end{bmatrix}, B = \begin{bmatrix} x & 0 \\ -x & -y \end{bmatrix}$$

Then A is a pseudo-EPM with 3 indeterminates and B is a pseudo-EPM with 2 indeterminates.

Nonsingular Pseudo-EPMs

A pseudo EPM is nonsingular over $\mathbb{F}$ (or $\mathbb{F}$-nonsingular) if its non-zero completions are nonsingular over $\mathbb{F}$. A nonsingular pseudo EPM with k indeterminates is a nonsingular vector space of dimension k.

Theorem

If $A(x_1, \cdots, x_k)$ is an $\mathbb{F}$-almost-nonsingular then $A(x_1, \cdots, x_{k-1}, 0)$ is an $\mathbb{F}$-nonsingular pseudo EPM.

Proof.

$$\begin{aligned} & \det A(a_1, \cdots, a_{k-1}, 0) = 0 \\ \Leftrightarrow \quad & a_1 = a_2 = \cdots = a_{k-1} = 0 \end{aligned}$$

□

Constructions of almost-nonsingular EPMs over $\mathbb{Q}$

Let $P(x_1, \cdots, x_k)$ be a $\mathbb{Q}$-nonsingular pseudo EPM. For an integer $m \geq 4$, we define $\Phi_m(P)$ to be the EPM of size $mn \times mn$ with indeterminates $x_1, \cdots, x_k, x$, in which the $m \times m$ blocks in the (i, j) position is given by

$$\begin{cases} T_m(x_t, x) & \text{if } P_{ij} = x_t \\ T_m(x, x_t) & \text{if } P_{ij} = -x_t \\ xJ_m & \text{if } P_{ij} = 0 \end{cases}$$

Theorem

Let $P(x_1, \cdots, x_k)$ be an $n \times n$ pseudo-EPM and let $m \geq 4$ be a positive integer. Then for any field $\mathbb{F}$, $\Phi_m(P)(x_1, \cdots, x_k, x)$ is an $\mathbb{F}$-almost-nonsingular EPM if and only if $P(x_1, \cdots, x_k)$ is $\mathbb{F}$-nonsingular.

Hence, it is reasonable to construct almost-nonsingular EPMs from nonsingular pseudo-EPMs.

Constructions of almost-nonsingular EPMs over $\mathbb{Q}$

If n is a power of 2 then $p(x) = x^n + 1$ is irreducible over $\mathbb{Q}$.

Constructions of almost-nonsingular EPMs over $\mathbb{Q}$

If n is a power of 2 then $p(x) = x^n + 1$ is irreducible over $\mathbb{Q}$. Since

$$p(x+1) = x^n + \sum_{i=1}^{n-1} \binom{n}{i} x^i + 2$$

If n is a power of 2 then $p(x) = x^n + 1$ is irreducible over $\mathbb{Q}$. Since

$$p(x+1) = x^n + \sum_{i=1}^{n-1} \binom{n}{i} x^i + 2$$

Now, let α be a root of $p(x)$ and $\mathbb{F} = \mathbb{Q}(\alpha)$ and let $\mathscr{B} = \{1, \alpha, \cdots, \alpha^{n-1}\}$ be a $\mathbb{Q}$-basis of $\mathbb{F}$.

Constructions of almost-nonsingular EPMs over $\mathbb{Q}$

If n is a power of 2 then $p(x) = x^n + 1$ is irreducible over $\mathbb{Q}$. Since

$$p(x+1) = x^n + \sum_{i=1}^{n-1} \binom{n}{i} x^i + 2$$

Now, let α be a root of $p(x)$ and $\mathbb{F} = \mathbb{Q}(\alpha)$ and let $\mathscr{B} = \{1, \alpha, \cdots, \alpha^{n-1}\}$ be a $\mathbb{Q}$-basis of $\mathbb{F}$. For every element a of $\mathbb{F}$, we write

$$\begin{array}{rccc} \sigma_a : & \mathbb{F} & \rightarrow & \mathbb{F} \\ & b & \mapsto & ab \end{array}$$

Constructions of almost-nonsingular EPMs over $\mathbb{Q}$

If n is a power of 2 then $p(x) = x^n + 1$ is irreducible over $\mathbb{Q}$. Since

$$p(x+1) = x^n + \sum_{i=1}^{n-1} \binom{n}{i} x^i + 2$$

Now, let α be a root of $p(x)$ and $\mathbb{F} = \mathbb{Q}(\alpha)$ and let $\mathscr{B} = \{1, \alpha, \cdots, \alpha^{n-1}\}$ be a $\mathbb{Q}$-basis of $\mathbb{F}$. For every element a of $\mathbb{F}$, we write

$$\begin{array}{rccc} \sigma_a : & \mathbb{F} & \to & \mathbb{F} \\ & b & \mapsto & ab \end{array}$$

Denote M_a the matrix of σ_a w.r.t. the basis $\mathscr{B}$.

Constructions of almost-nonsingular EPMs over $\mathbb{Q}$

If n is a power of 2 then $p(x) = x^n + 1$ is irreducible over $\mathbb{Q}$. Since

$$p(x+1) = x^n + \sum_{i=1}^{n-1} \binom{n}{i} x^i + 2$$

Now, let α be a root of $p(x)$ and $\mathbb{F} = \mathbb{Q}(\alpha)$ and let $\mathscr{B} = \{1, \alpha, \cdots, \alpha^{n-1}\}$ be a $\mathbb{Q}$-basis of $\mathbb{F}$. For every element a of $\mathbb{F}$, we write

$$\begin{array}{rccc} \sigma_a : & \mathbb{F} & \to & \mathbb{F} \\ & b & \mapsto & ab \end{array}$$

Denote M_a the matrix of σ_a w.r.t. the basis $\mathscr{B}$. Then

$$\left\{M_1, M_\alpha, \cdots, M_{\alpha^{n-1}}\right\}$$

is an $\mathbb{Q}$-linear space of dimension n and no two of them have the non-zero entries at the same position.

Constructions of almost-nonsingular EPMs over $\mathbb{Q}$

Therefore,

$$P := x_1 M_1 + x_2 M_\alpha + \cdots + x_n M_{\alpha^{n-1}}$$

is a pseudo-epm which is $\mathbb{Q}$-nonsingular.

Constructions of almost-nonsingular EPMs over $\mathbb{Q}$

Therefore,

$$P := x_1 M_1 + x_2 M_\alpha + \cdots + x_n M_{\alpha^{n-1}}$$

is a pseudo-epm which is $\mathbb{Q}$-nonsingular.

Theorem

If n is a power of 2, there exists a $\mathbb{Q}$-nonsingular pseudo-epm of size $n \times n$ which has n indeterminates. Hence, for every $m \geq 4$, there exists a $\mathbb{Q}$-almost-nonsingular EPM of size $mn \times mn$ which has $n+1$ indeterminates.

$$\tau_{\mathbb{Q}}(m \cdot 2^k) \geq 2^k + 1 \text{ for every integer } m \geq 4, k \geq 0.$$

Constructions of almost-nonsingular EPMs over $\mathbb{Q}$

Similarly $x^n - x - 1$ is irreducible over $\mathbb{Q}$ for every $n \geq 2$.

$$x_1 M_1 + \sum_{i=1,\, i \text{ odd}}^{n-1} x_i M_{\alpha^i}$$

is a nonsingular pseudo-EPM over $\mathbb{Q}$.

Constructions of almost-nonsingular EPMs over $\mathbb{Q}$

Similarly $x^n - x - 1$ is irreducible over $\mathbb{Q}$ for every $n \geq 2$.

$$x_1 M_1 + \sum_{i=1, i \text{ odd}}^{n-1} x_i M_{\alpha^i}$$

is a nonsingular pseudo-EPM over $\mathbb{Q}$.

Theorem

Let n be a positive integer. Then for every $m \geq 4$, there exists a $\mathbb{Q}$-almost-nonsingular EPM of size $mn \times mn$ which has $\lceil \frac{n}{2} \rceil + 1$ indeterminates.

$$\tau_{\mathbb{Q}}(mn) \geq \left\lceil \frac{n}{2} \right\rceil + 1$$

Constructions of almost-nonsingular EPMs over $\mathbb{Q}$

If $n = n_1 + n_2$ then

$$\tau_{\mathbb{F}}(n) \geq \min\{\tau_{\mathbb{Q}}(n_1), \tau_{\mathbb{Q}}(n_2)\}$$

On the other hand, for $n \geq 12$, there exist non-negative integers s, t such that $n = 4s + 5t$, where $|s - t| \leq 4$. Hence,

$$\min\left\{\frac{s}{2}, \frac{t}{2}\right\} \geq \frac{n-20}{18} = \frac{n-2}{18} - 1$$

Theorem

Let n be a positive integer. Then

$$\tau_{\mathbb{Q}}(n) \geq \left\lceil \frac{n-2}{18} \right\rceil$$

THANK YOU!