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Stochastic Zero-Time Discontinuity Mappings

EOGHAN J. STAUNTON,
PETRI T. PIIROINEN

27TH SEPTEMBER 2019

Motivation

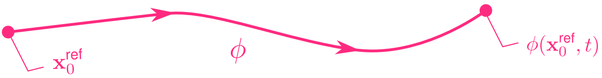

Figure: Smooth linearisation.

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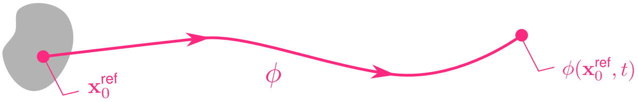

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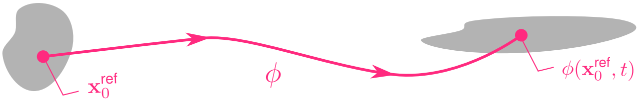

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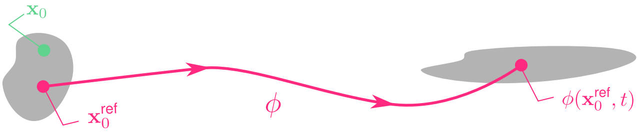

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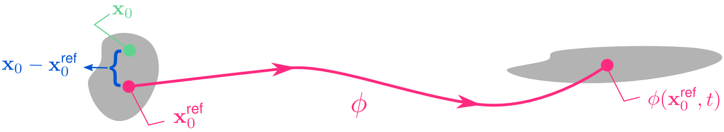

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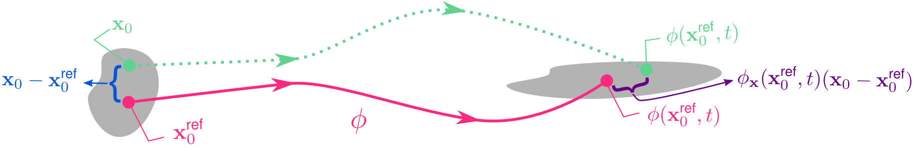

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Transversal Crossings

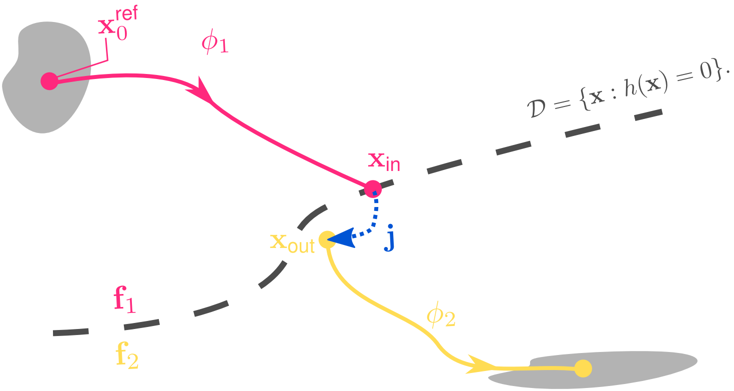

Figure: A nonsmooth dynamical system.

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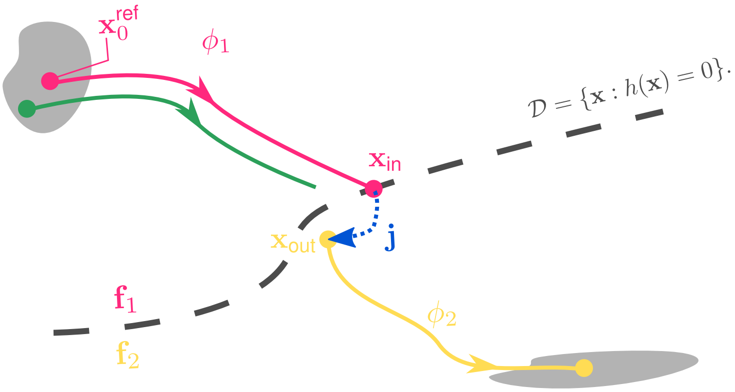

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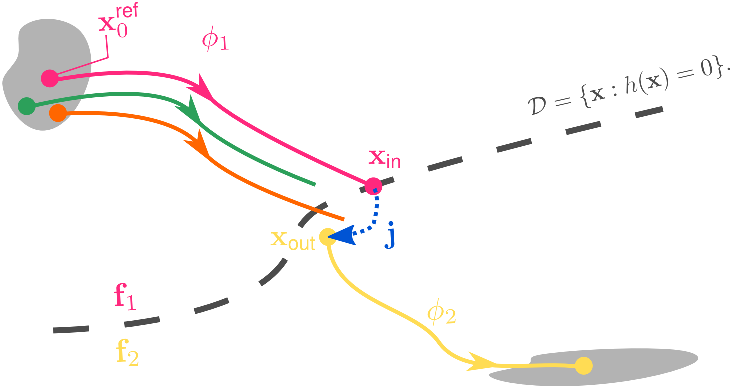

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Constructing the Zero-Time Discontinuity Mapping

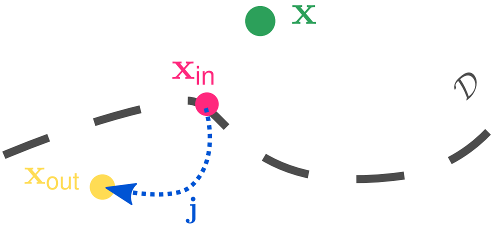

Figure: Constructing the ZDM.

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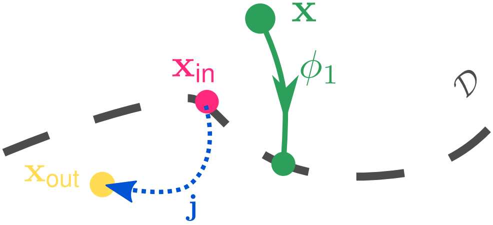

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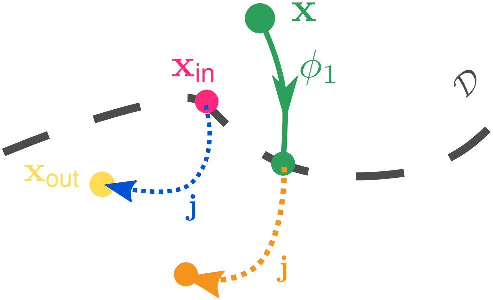

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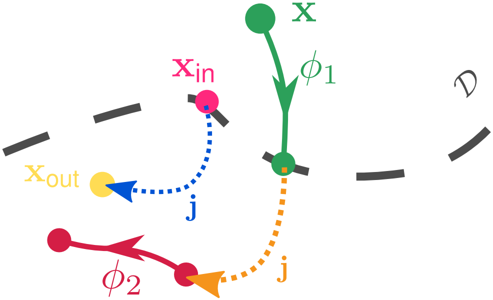

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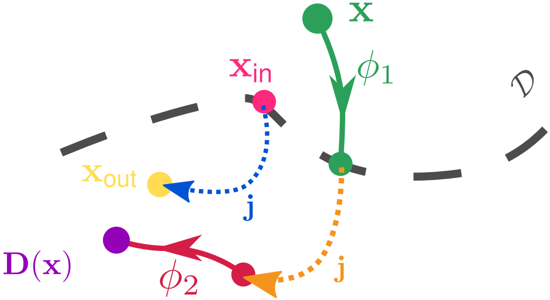

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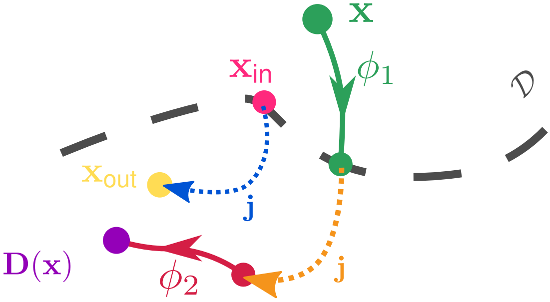

Figure: Constructing the ZDM.

We can now write

$$\phi(\mathbf{x}_0, T) = \phi_2(\mathbf{D}(\phi_1(\mathbf{x}_0, t_{\text{ref}})), T - t_{\text{ref}}),$$

where the ZDM

$$\mathbf{D}(\mathbf{x}) = \phi_2(\mathbf{j}(\phi_1(\mathbf{x}, t(\mathbf{x}))), -t(\mathbf{x}))$$

takes a point in a neighbourhood of $\mathbf{x}_{\text{in}}$ and maps it to a point in a neighbourhood of $\mathbf{x}_{\text{out}}$.

Grazing Interactions

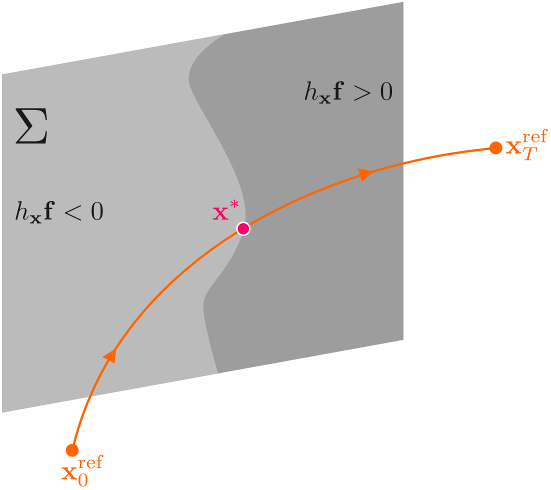

Figure: A grazing interaction in a hybrid system.

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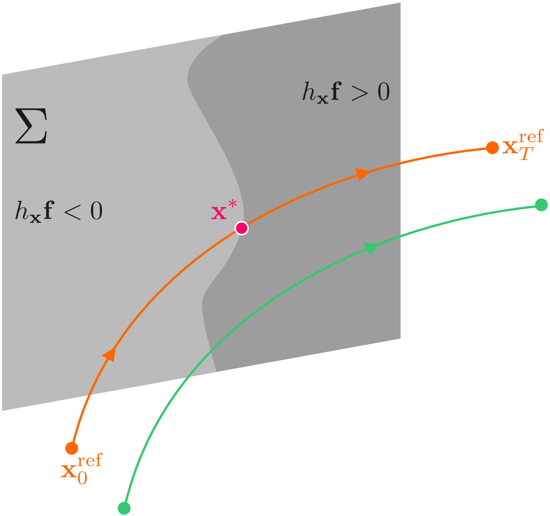

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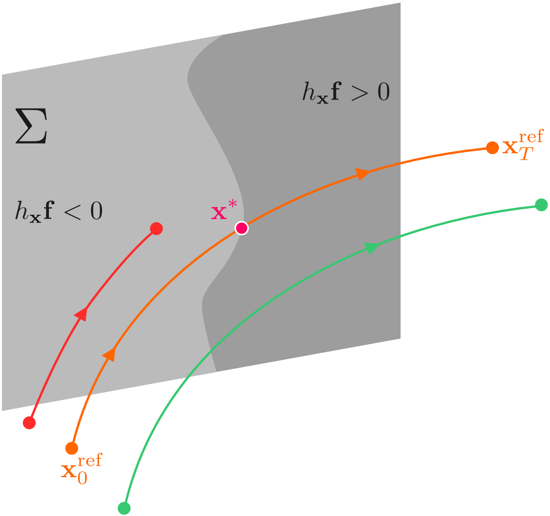

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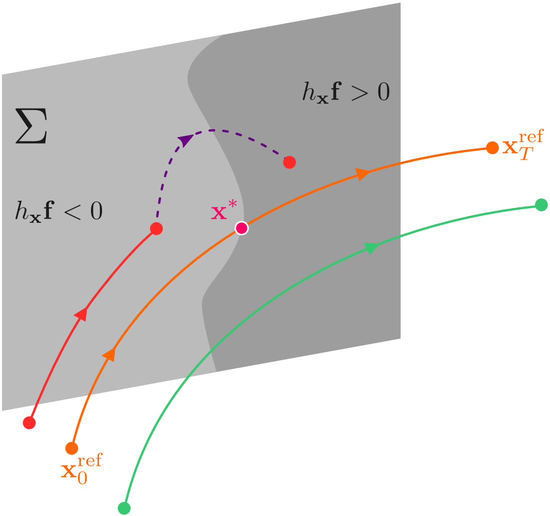

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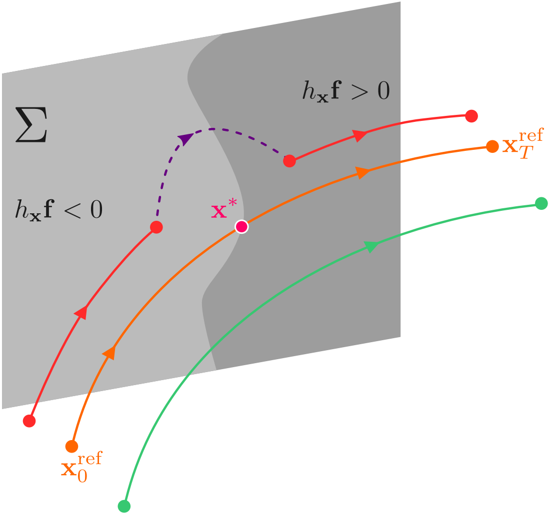

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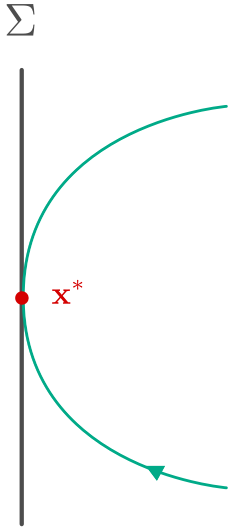

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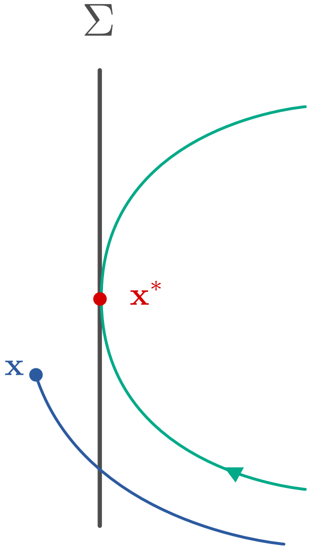

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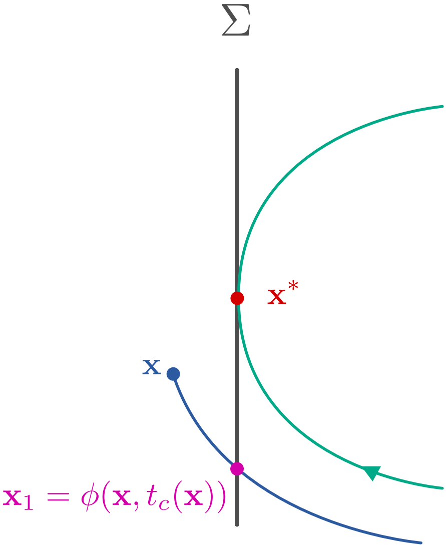

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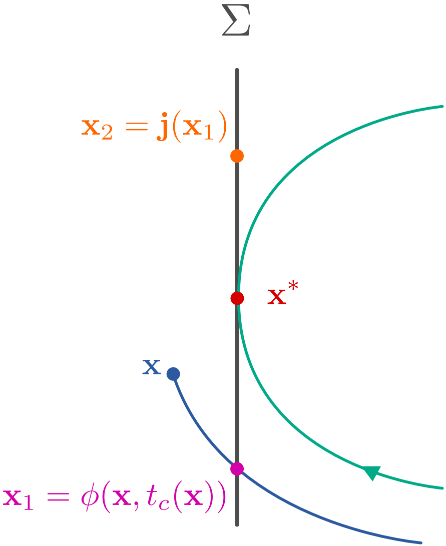

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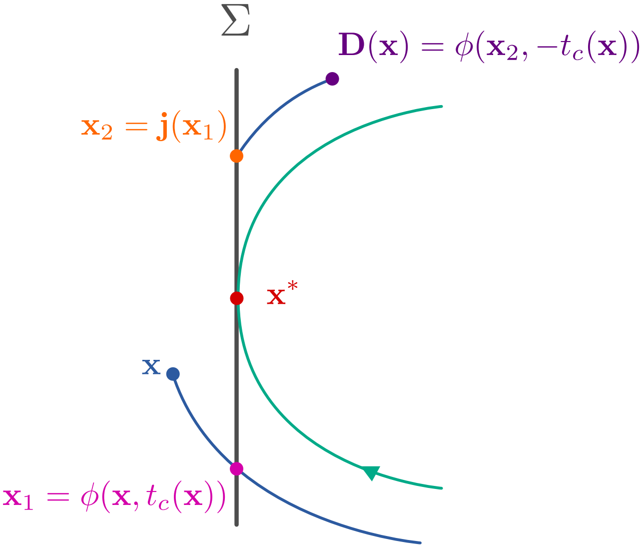

Figure: Constructing the grazing ZDM.

Types of Noise

Figure: A rugged boundary.

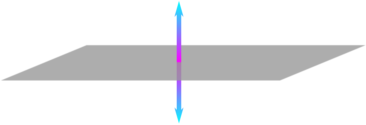

Figure: An oscillating boundary.

Rugged boundaries are suitable for modelling situations where the small-scale structure of the boundary is uncertain.

$$\tilde{\Sigma} = \{\mathbf{x} : \tilde{h}(\mathbf{x}, t) = 0\},$$
$$\tilde{h}(\mathbf{x}, t) = h(\mathbf{x}, t) - \chi(\mathbf{x}).$$

Oscillating boundaries are suitable for modelling situations where the boundary has small uncertain oscillations about a known mean.

$$\tilde{\Sigma} = \{\mathbf{x} : \tilde{h}(\mathbf{x}, t) = 0\},$$
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Noise Requirements

In order for us to carry out our analysis we require that the stochastic processes $\chi(\mathbf{x})$ and $P(t)$ describing the stochastic components of the discontinuity boundaries

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- are at least once differentiable,
- are of small amplitude,
- are mean reverting,
- have mean 0.

Transversal SZDM

For transversal crossings we base our approximation on linearisation about the corresponding trajectory in the deterministic system, taking

$$\Delta t_{\mathsf{ref}} = \mathcal{P} / \left(h_{\mathbf{x}}(\mathbf{x}_{\mathsf{in}}, t_{\mathsf{ref}}) \mathbf{f}_{\mathsf{in}} + h_t(\mathbf{x}_{\mathsf{in}}, t_{\mathsf{ref}}) - \mathcal{V} \right),$$

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where $\mathcal{P} = \chi(\mathbf{x}_{\text{in}})$, $\mathcal{V} = \chi_{\mathbf{x}}(\mathbf{x}_{\text{in}})\mathbf{f}_{\text{in}}$ in the rugged boundary case and $\mathcal{P} = P(t_{\text{ref}})$, $\mathcal{V} = V(t_{\text{ref}})$ in the case of an oscillating boundary.

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We then find that

$$\begin{aligned}
\phi(\mathbf{x}_0, T) \quad - \quad & \phi(\mathbf{x}_0^{\text{ref}}, T) \approx \\
& \phi_{\mathbf{x}}(\mathbf{x}_0^{\text{ref}}, T)(\mathbf{x}_0 - \mathbf{x}_0^{\text{ref}}) + \phi_{2,\mathbf{x}}(\mathbf{x}_{\text{out}}, T - t_{\text{ref}})\mathcal{N}(\mathbf{x}_{\text{in}}, t_{\text{ref}})\Delta t_{\text{ref}} \\
& + \phi_{2,\mathbf{x}}(\mathbf{x}_{\text{out}}, T - t_{\text{ref}})\mathcal{J}(\mathbf{x}_{\text{in}}, t_{\text{ref}}),
\end{aligned}$$

where

$$\phi_{\mathbf{x}}(\mathbf{x}_0^{\text{ref}}, T) = \phi_{2,\mathbf{x}}(\mathbf{x}_{\text{out}}, T - t_{\text{ref}})\tilde{\mathbf{D}}_{\mathbf{x}}(\mathbf{x}_{\text{in}})\phi_{1,\mathbf{x}}(\hat{\mathbf{x}}_{\text{in}}, t_{\text{ref}}),$$

$$\mathcal{N}(\mathbf{x}_{\text{in}}, t_{\text{ref}}) = \mathbf{j}_{\mathbf{x}}(\mathbf{x}_{\text{in}}, t_{\text{ref}})\mathbf{f}_{\text{in}} + \mathbf{j}_t(\mathbf{x}_{\text{in}}, t_{\text{ref}})) - \mathbf{f}_{\text{out}},$$

and

$$\mathcal{J}(\mathbf{x}_{\text{in}}, t_{\text{ref}}) = \tilde{\mathbf{j}}(\mathbf{x}_{\text{in}} | \mathcal{P} = 0) - \mathbf{j}(\mathbf{x}_{\text{in}}).$$

Example - Boucing a ball on a rugged oscillating floor.

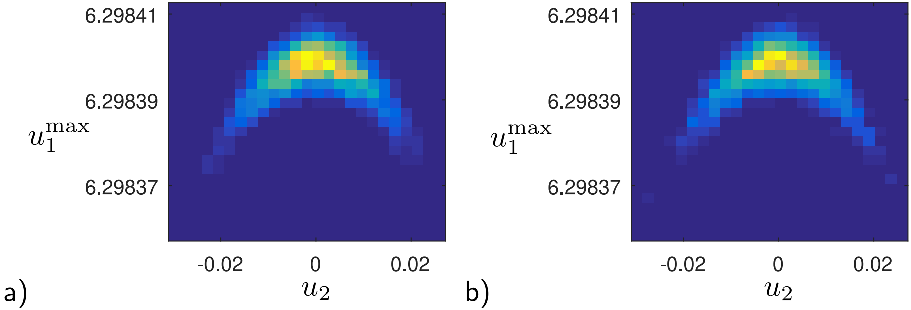

Figure: Heatmaps of the distribution of the maximum height attained by the bouncing ball and its corresponding horizontal position after one bounce on the rugged surface given by a) full simulation of the system b) linear approximation.

SZDMs for Higher-Order Discontinuities

When the vector field is $\mathcal{C}^{n-1}$ for $n \geq 1$ but has higher-order discontinuities the are are no linear effects so one must consider higher order approximations to capture the effects of noise and the crossing of a discontinuity boundary.

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$$\tilde{\mathbf{D}}(\mathbf{x}) \approx \mathbf{x} + \frac{g(\mathbf{x}^*)}{(n+1) h_{\mathbf{x}}(\mathbf{x}_{\text{in}}) \mathbf{f}_{\text{in}}} \left(h(\mathbf{x})^{n+1} - \left(\frac{\mathcal{P} h_{\mathbf{x}}(\mathbf{x}_{\text{in}}) \mathbf{f}_{\text{in}} - h(\mathbf{x}) \mathcal{V}}{h_{\mathbf{x}}(\mathbf{x}_{\text{in}}) \mathbf{f}_{\text{in}} - \mathcal{V}} \right)^{n+1} \right),$$

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where

$$g(\mathbf{x}) = \sum_{j=0}^{\infty} \frac{h^j}{(j+n)!} \frac{\partial^{j+n}}{\partial h^{j+n}} (\mathbf{f}_2 - \mathbf{f}_1)|_{h=0},$$

$\mathcal{P} = \chi(\mathbf{x}_{\text{in}})$, $\mathcal{V} = \chi_{\mathbf{x}}(\mathbf{x}_{\text{in}})\mathbf{f}_{\text{in}}$ in the rugged boundary case and $\mathcal{P} = P(t_{\text{ref}})$, $\mathcal{V} = V(t_{\text{ref}})$ in the case of an oscillating boundary.

Example - The Chua Circuit

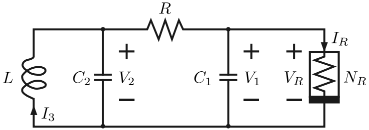

Figure: The Chua circuit.

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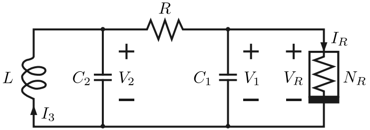

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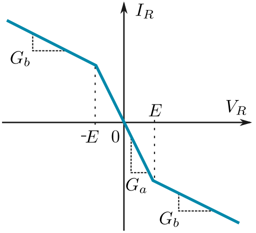

Figure: V-I characteristic.

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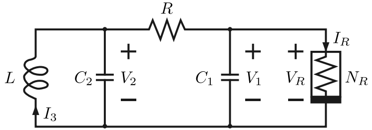

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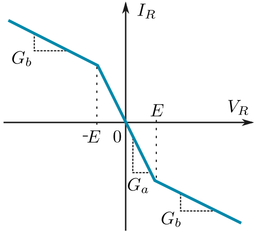

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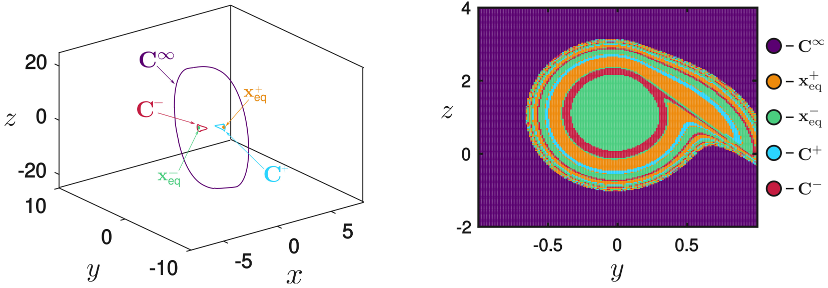

Figure: Coexisting attractors in the Chua circuit.

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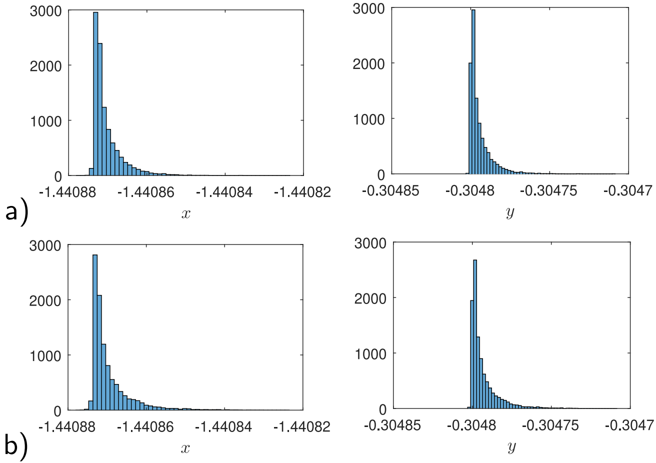

Figure: The Chua circuit with oscillating boundaries. The results of full numerical-simulation are shown in a) and the approximations obtained by using the SZDM in place of boundary interactions are shown in b).

Grazing SZDM

In the case of a grazing interaction we cannot linearise in the same way as we considered in the case of a transversal crossing. Instead we consider second-order approximations about the point and time where the deterministic component of $\tilde{h}$ (which we denote h) reaches its minimum value.

Grazing SZDM

In the case of a grazing interaction we cannot linearise in the same way as we considered in the case of a transversal crossing. Instead we consider second-order approximations about the point and time where the deterministic component of $\tilde{h}$ (which we denote h) reaches its minimum value. We find that $\tilde{\mathbf{D}}(\mathbf{x}) = \mathbf{x} + \tilde{\Delta}\mathbf{x}$

$$\tilde{\Delta}\mathbf{x} \approx \sqrt{\left(-h_{\mathbf{x}}(\mathbf{x}^*)\mathbf{x} + \mathcal{P} + \mathcal{V}\frac{h_{\mathbf{x}}\mathbf{f}}{(h_{\mathbf{xx}}\mathbf{f} + h_{\mathbf{x}}\mathbf{f}_{\mathbf{x}})\mathbf{f}} + \frac{\mathcal{V}^2}{2(A_g - \mathcal{A})}\right) 2(A_g - \mathcal{A})\xi},$$

where $\mathcal{P} = \chi$, $\mathcal{V} = \chi_{\mathbf{x}}\mathbf{f}$, $\mathcal{A} = (\chi_{\mathbf{xx}}\mathbf{f} + \chi_{\mathbf{x}}\mathbf{f}_{\mathbf{x}})\mathbf{f}$ in the rugged boundary case and $\mathcal{P} = P$, $\mathcal{V} = V$, $\mathcal{A} = A$ in the case of an oscillating boundary.

Example - A grazing impact oscillator

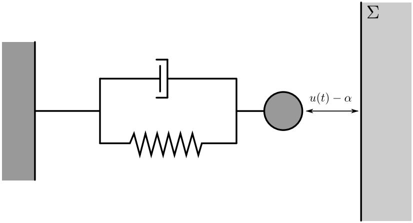

Figure: Schematic of a one-degree-of-freedom impact oscillator.

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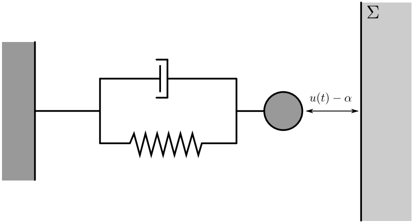

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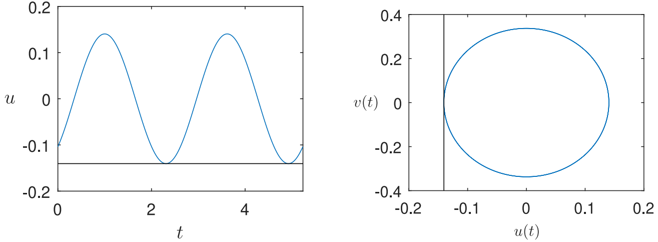

Figure: A sample grazing orbit.

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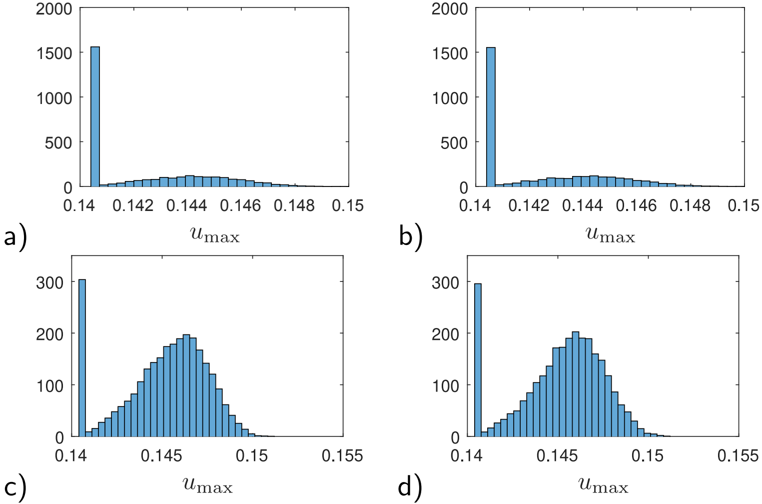

Figure: Histograms of the pdf of the maximum amplitude attained by the impact oscillator by a), c), full simulation of the system and b), d) approximation using the SZDM. a), b) $\varepsilon = 0$, c), d) $\varepsilon = 0.00005$.

M A Aizerman and F R Gantmacher, *Determination of stability by linear approximation of a periodic solution of a system of differential equations with discontinuous right-hand sides*, The Quarterly Journal of Mechanics and Applied Mathematics **11** (1958), no. 4, 385–398.

Mats H Fredriksson and Arne B Nordmark, *Bifurcations caused by grazing incidence in many degrees of freedom impact oscillators*, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, vol. 453, The Royal Society, 1997, pp. 1261–1276.

Arne B Nordmark, *Discontinuity mappings for vector fields with higher order continuity*, Dynamical systems **17** (2002), no. 4, 359–376.

Eoghan J Staunton and Petri T Piiroinen, *Discontinuity mappings for stochastic nonsmooth systems*, Submitted (2019).

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