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Boundary Noise in Chua's Circuit

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Chua's Circuit

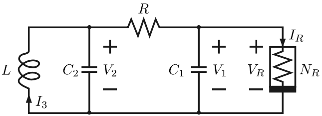


Figure: Chua's Circuit

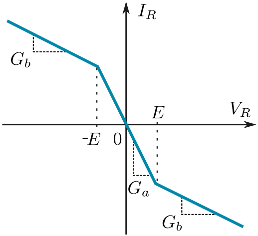


Figure: The $V - I$ characteristic of Chua's Diode.

- Created with the aim of being the simplest autonomous circuit capable of generating chaos [Mat84, Chu92].
- First physical system for which the presence of chaos was shown experimentally, numerically and mathematically [CKM86].
- Contains four linear elements and one nonlinear resistor known as a *Chua's diode*.
- Easily and cheaply constructed using stadard electronic components [Ken92].

System equations

The system equations of Chua's circuit are

$$\begin{aligned} \frac{dV_1}{dt} &= \frac{1}{C_1}(G(V_2 - V_1) - f(V_1)), \quad \frac{dV_2}{dt} = \frac{1}{C_2}(G(V_1 - V_2) + I_3) \\ \frac{dI_3}{dt} &= -\frac{1}{L}(V_2 + R_0 I_3), \end{aligned} \tag{1}$$

where

$$G = \frac{1}{R} \text{ and } f(V_1) = G_b V_1 + \frac{1}{2}(G_a - G_b)(|V_1 + E| - |V_1 - E|). \tag{2}$$

Nondimensionalising gives

$$\begin{aligned} \frac{dx}{dt'} &= \alpha(y - x - g(x)), \quad \frac{dy}{dt'} = x - y + z, \\ \frac{dz}{dt'} &= -(\beta y + \gamma z), \end{aligned} \tag{3}$$

where

$$g(x) = m_1 x + \frac{1}{2}(m_0 - m_1)(|x + 1| - |x - 1|). \tag{4}$$

Complicated Dynamics

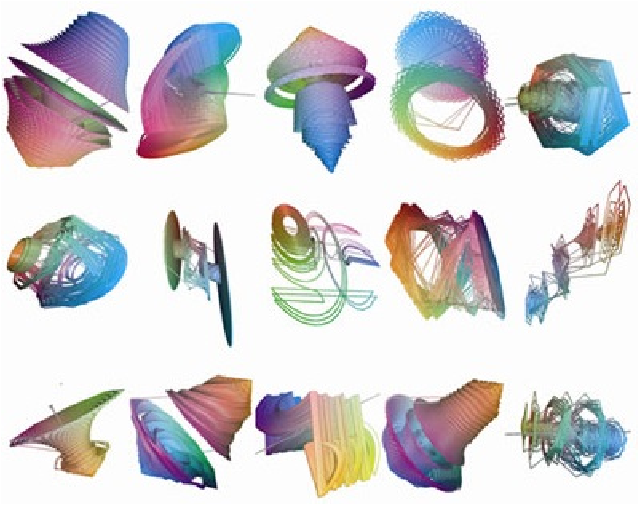
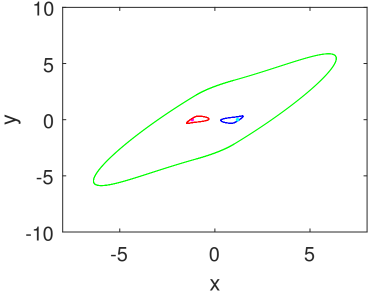


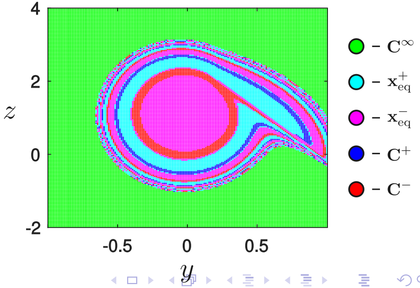
Figure: A Zoo of Attractors Produced by Chua's Circuit [BP08]

Hidden and Self-Excited Attractors

- **Hidden attractors:** have basins of attraction that do not intersect with small neighborhoods of equilibria.
- **Self-excited attractors**: Can be found by following trajectories from the neighbourhoods of unstable equilibria until the end of a transient process [LK13].

Most classical attractors are self-excited attractors and easily found. On the other hand the search for hidden attractors can be difficult. The first chaotic hidden attractor was found in Chua's circuit in 2010 [KLV10].





Noisy Saltation

In order to deal the with stochastically moving boundaries when linearising we generalise the concept of a *saltation matrix*, the matrix that allows us to deal with nonsmoothness when linearising deterministic systems.
We extend the state space, such that the state vector is $\tilde{\mathbf{x}} = (\mathbf{x}, t, \Delta t_{\text{ref}})^T$. Here Δt_{ref} is the random quantity which represents the difference in the hitting time of the reference trajectory in the stochastic system compared to the corresponding deterministic system.
We calculate the saltation matrix in this extended state space before projecting back. As a result, in the original state space we find that

$$\phi(\mathbf{x}_0, t) - \phi(\hat{\mathbf{x}}_0^{\text{ref}}, t) \approx \phi_{\mathbf{x}}(\hat{\mathbf{x}}_0^{\text{ref}}, t)(\mathbf{x}_0 - \hat{\mathbf{x}}_0^{\text{ref}}) + \phi_{2,\mathbf{x}}(\hat{\mathbf{x}}_{\text{out}}, t - \hat{t}_{\text{ref}})(\hat{\mathbf{f}}_{\text{in}} - \hat{\mathbf{f}}_{\text{out}})\Delta t_{\text{ref}}, \tag{5}$$

where

$$\phi_{\mathbf{x}}(\hat{\mathbf{x}}_0^{\text{ref}}, t) = \phi_{2,\mathbf{x}}(\hat{\mathbf{x}}_{\text{out}}, t - \hat{t}_{\text{ref}})\mathbf{D}^*_{\mathbf{x}}(\hat{\mathbf{x}}_{\text{in}})\phi_{1,\mathbf{x}}(\hat{\mathbf{x}}_{\text{in}}, \hat{t}_{\text{ref}}) \tag{6}$$

and

$$\mathbf{D}^*_{\mathbf{x}}(\hat{\mathbf{x}}_{\text{in}}) = \mathbf{I} + \frac{(\hat{\mathbf{f}}_{\text{out}} - \hat{\mathbf{f}}_{\text{in}})\hat{h}_{\mathbf{x}}(\hat{\mathbf{x}}_{\text{in}})}{\hat{h}_{\mathbf{x}}(\hat{\mathbf{x}}_{\text{in}})\hat{\mathbf{f}}_{\text{in}} - \hat{v}(\hat{t}_{\text{ref}}) - V(\hat{t}_{\text{ref}} | P(\hat{t}_{\text{ref}}) = 0)} \tag{7}$$

A Discontinuous Model

We note that in the case of continuous piecewise-smooth systems that $\hat{\mathbf{f}}_{\text{in}} = \hat{\mathbf{f}}_{\text{out}}$ and so (5) collapses to

$$\phi(\mathbf{x}_0, t) - \phi(\hat{\mathbf{x}}_0^{\text{ref}}, t) \approx \phi_{2,\mathbf{x}}(\hat{\mathbf{x}}_{\text{out}}, t - \hat{t}_{\text{ref}})\phi_{1,\mathbf{x}}(\hat{\mathbf{x}}_{\text{in}}, \hat{t}_{\text{ref}})(\mathbf{x}_0 - \hat{\mathbf{x}}_0^{\text{ref}}) \quad (8)$$

meaning that neither the noise nor the nonsmoothness has a linear effect. As a result we will consider an discontinuous model of Chua's circuit where the function describing the $V - I$ characteristic of Chua's diode given in (4) is replaced by a discontinuous one

$$g(x) = \begin{cases} m_1 x + m_1 - m_0 & \text{if } x < -1, \\ (m_0 - \epsilon)x & \text{if } |x| \leq 1, \\ m_1 x + m_0 - m_1 & \text{if } x > 1. \end{cases} \quad (9)$$

A Discontinuous Model

Provided the magnitude of ϵ is not too large the hidden attractors shown in Figure 4 continue to exist and can be easily found by numerical continuation.

They are destroyed in saddle-bifurcations if the magnitude of ϵ grows too large.

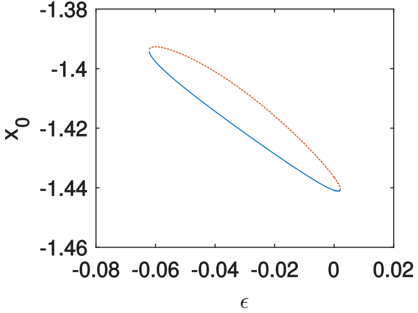


Figure: Bifurcation diagram showing the saddle bifurcations of $\mathbf{C}^-$ as the magnitude of ϵ grows. Here $\alpha = 8.4$, $\beta = 12$, $\gamma = -0.005$, $m_0 = -1.2$ and $m_1 = 0.145$.

Steady-State Distributions

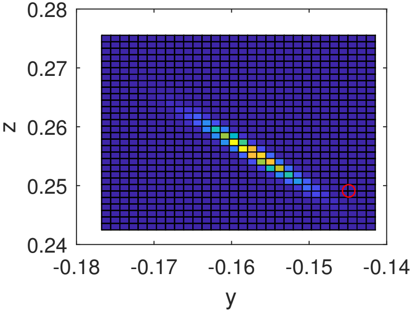


Figure: Steady state distribution of orbit errors on the discontinuity boundary $\mathcal{D}^-$ for trajectories with initial condition on the periodic orbit $\mathbf{C}^-$.

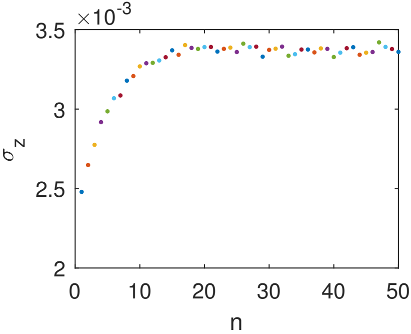


Figure: Convergence of σ_z to its steady state value for the distribution shown on the left.

The Next Steps

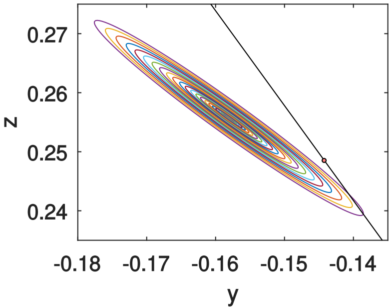


Figure: 5 standard deviation ellipses for increasing values of noise amplitude.

- Use our method to predict changes in qualitative behaviour.
  - Destruction of attractors
  - Merging of attractors
  - Switching/Flickering
  - Multi/Monostability
- Interaction of linearised distributions with features of the deterministic system.

- Generalise our method
  - Second order terms for continuous systems
  - Non-identity boundary mappings
  - Dealing with non-transversal intersections

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