Alternating Signed Bipartite Graph Colourings

Cian O'Brien
Rachel Quinlan and Kevin Jennings

Postgraduate Modelling Research Group
National University of Ireland, Galway

c.obrien40@nuigalway.ie

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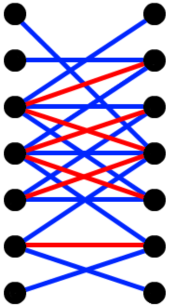

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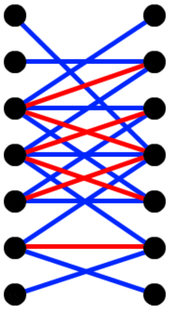

We say that G is *configurable*.

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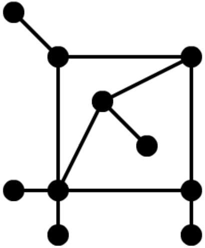

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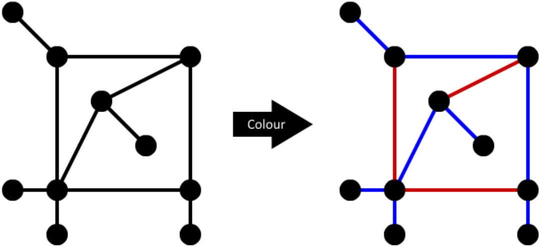

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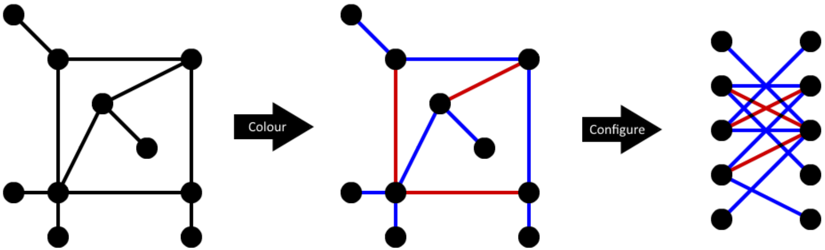

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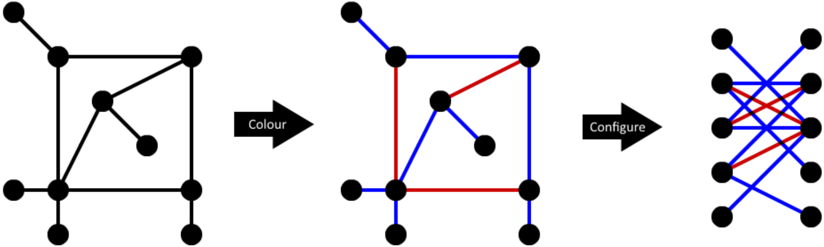

- We say a graph G has a *feasible colouring* c of its edges if $deg^B(v) - deg^R(v) = 1, \forall v \in V(G)$.

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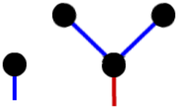

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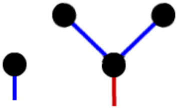

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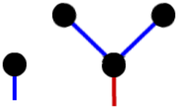

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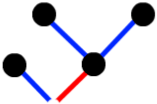

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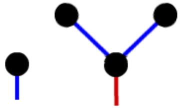

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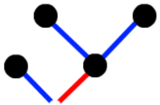

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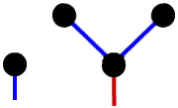

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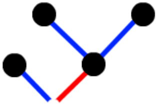

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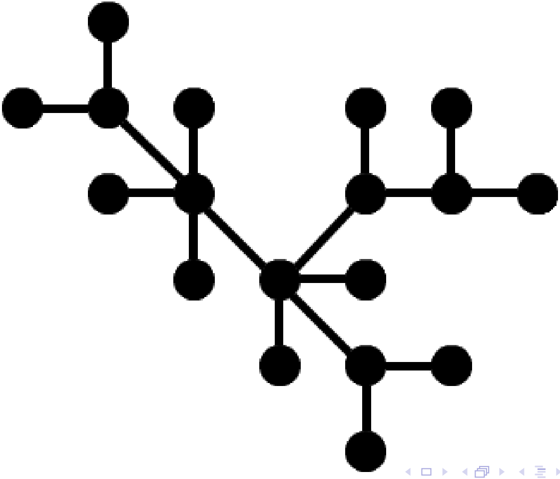

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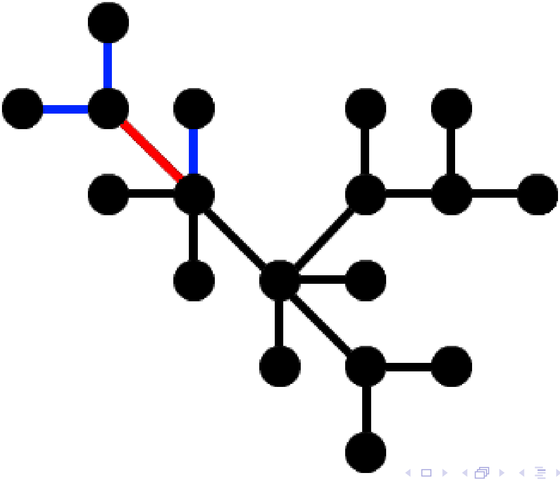

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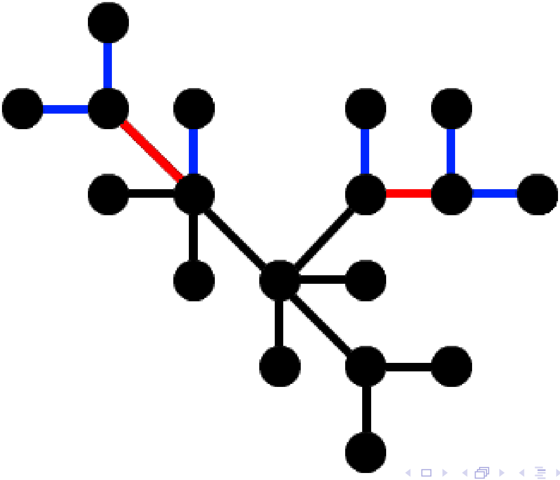

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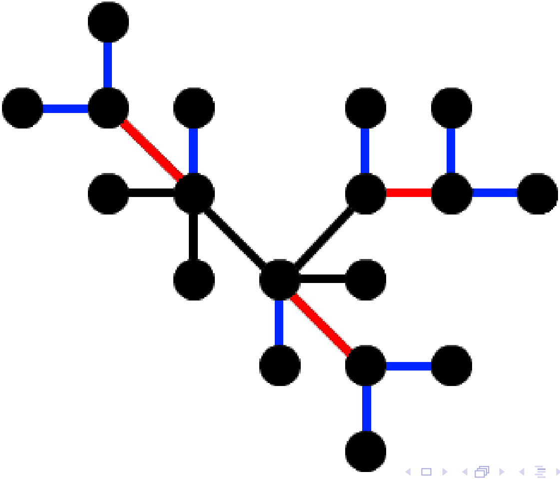

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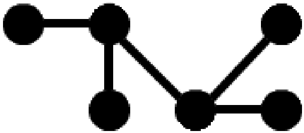

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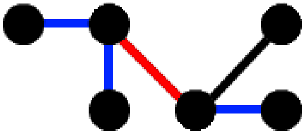

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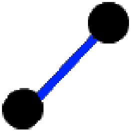

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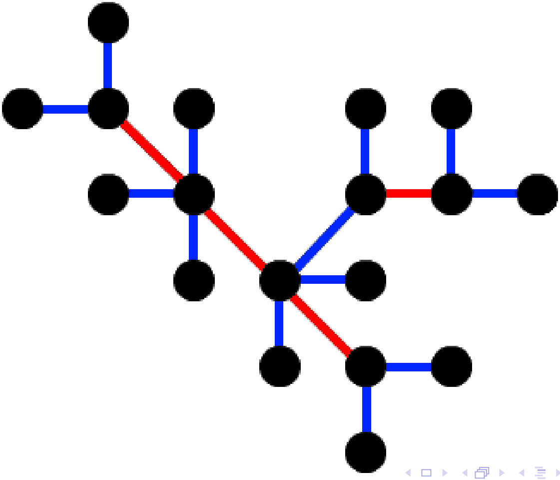

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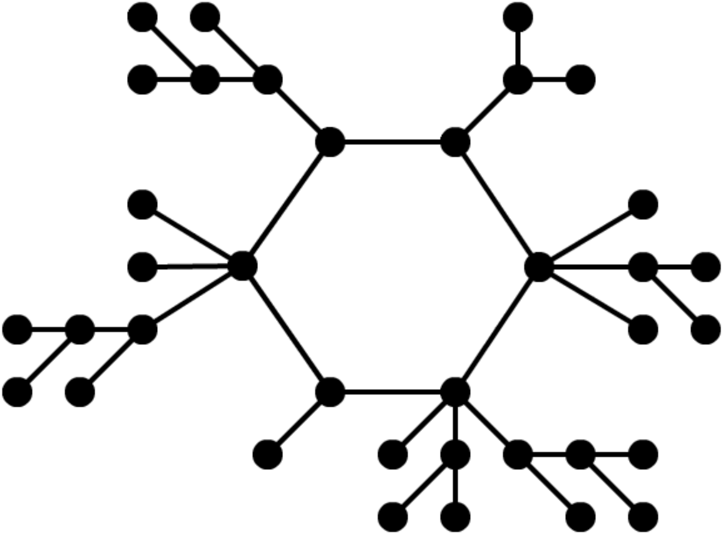

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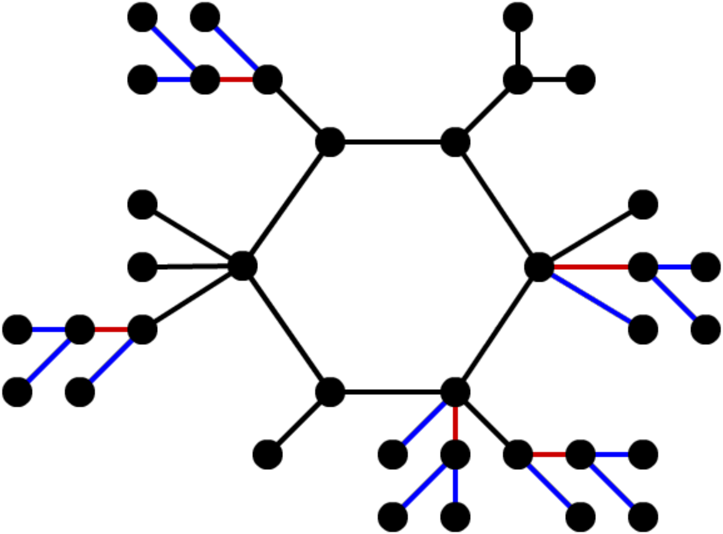

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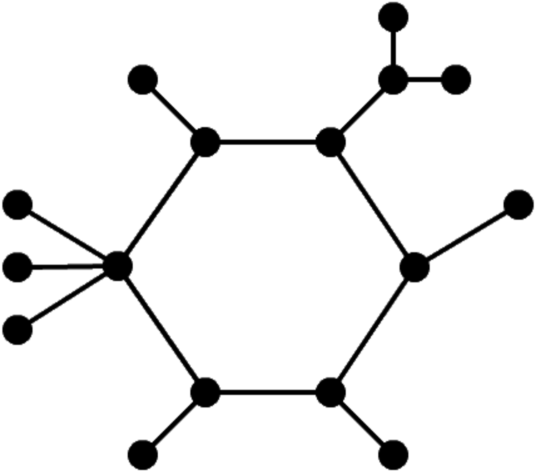

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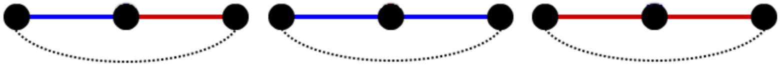

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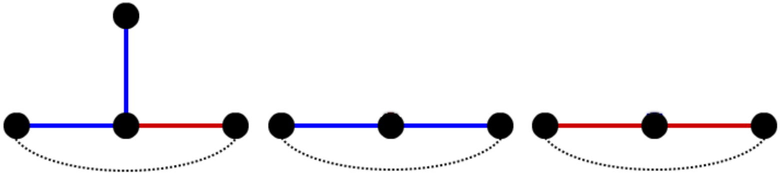

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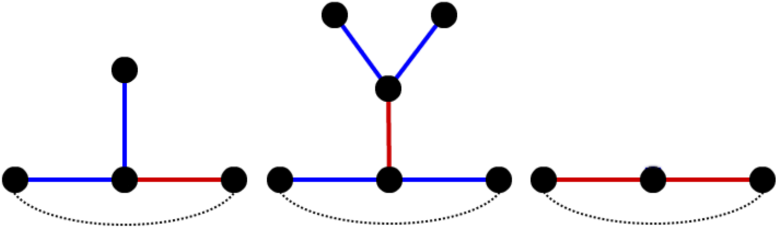

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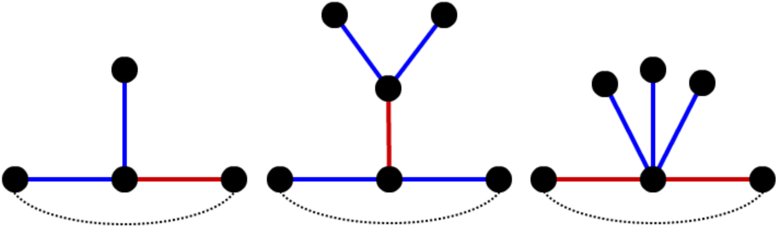

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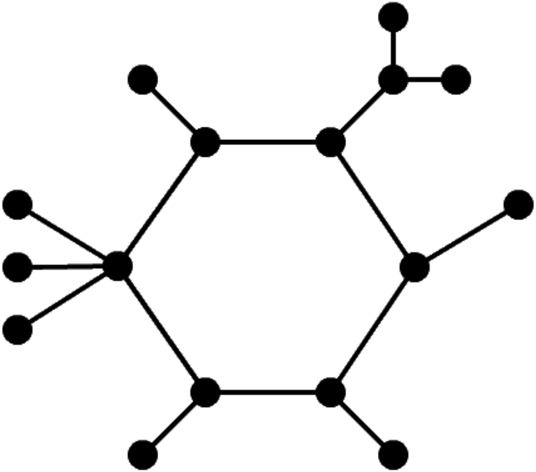

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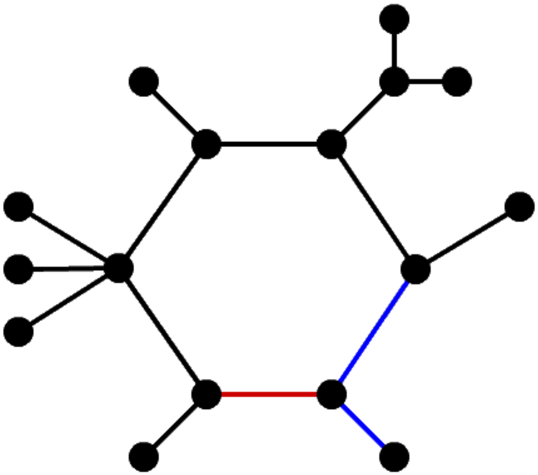

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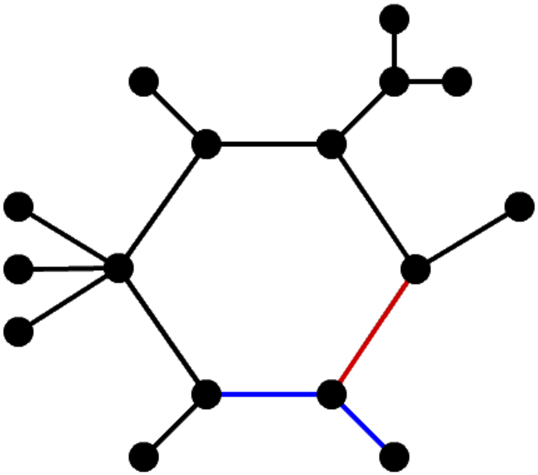

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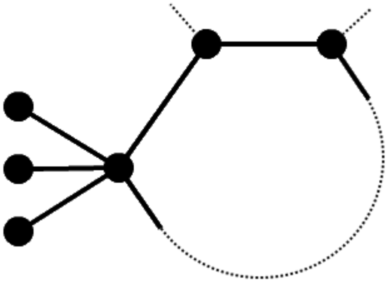

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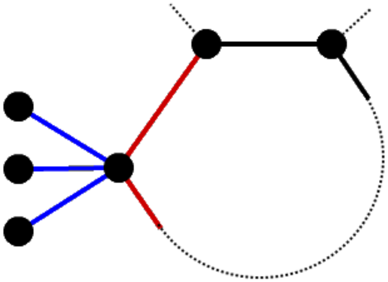

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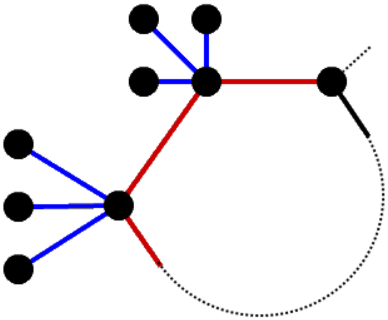

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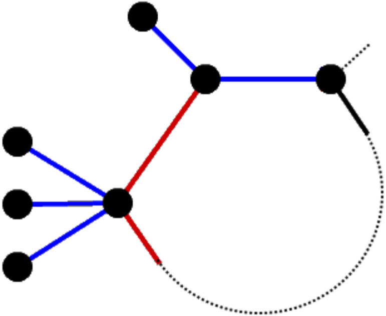

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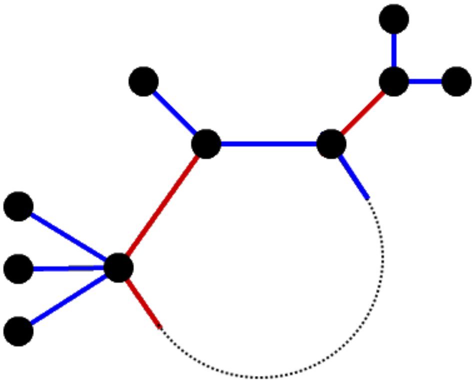

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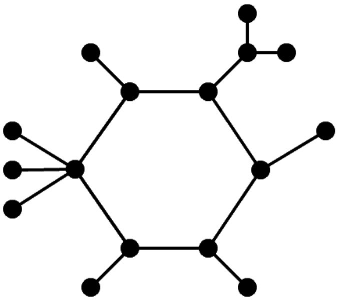

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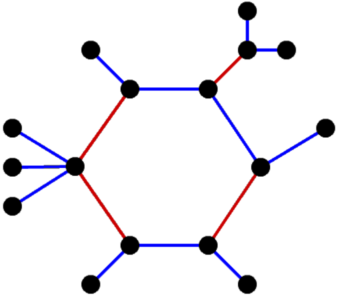

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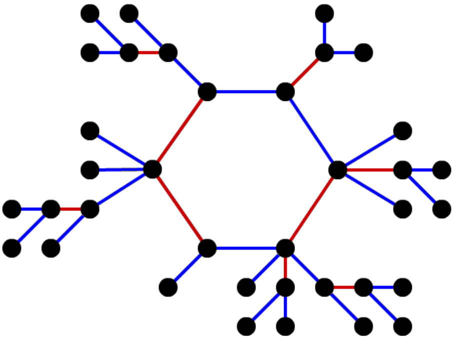

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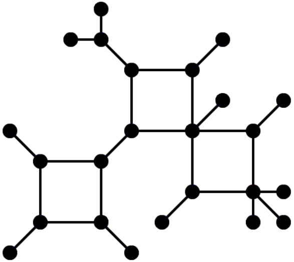

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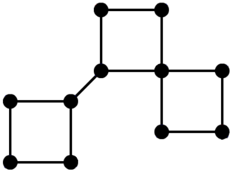

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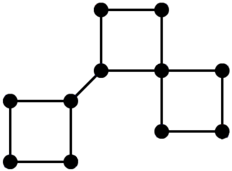

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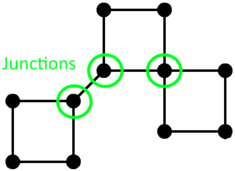

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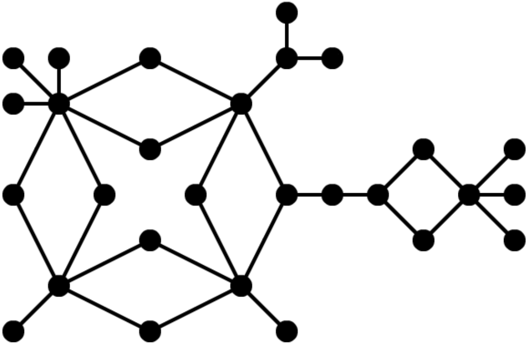

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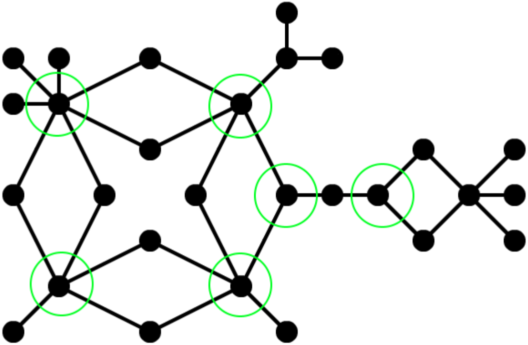

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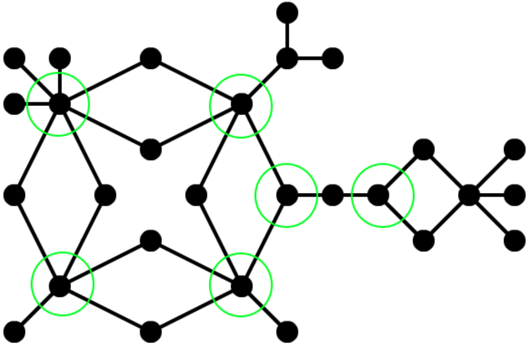

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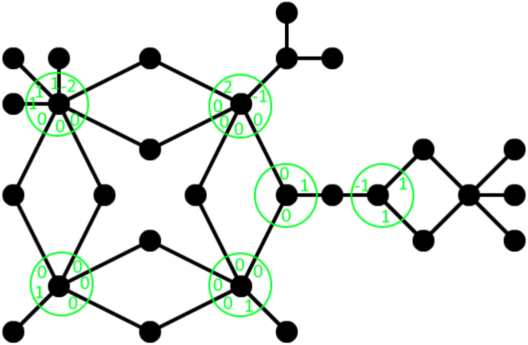

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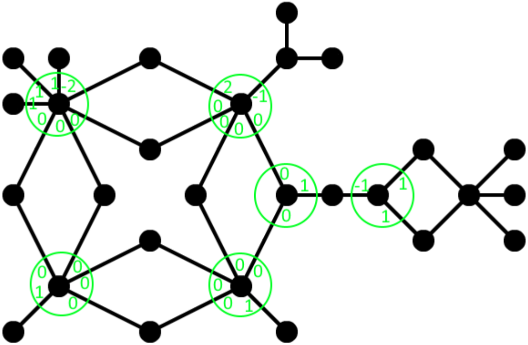

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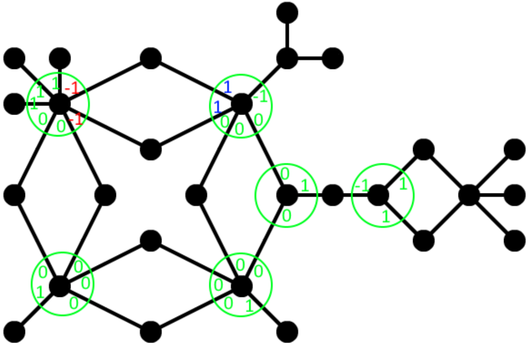

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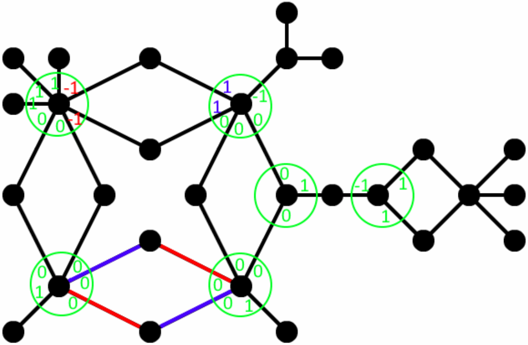

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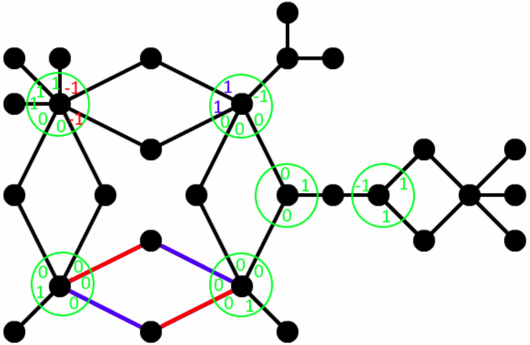

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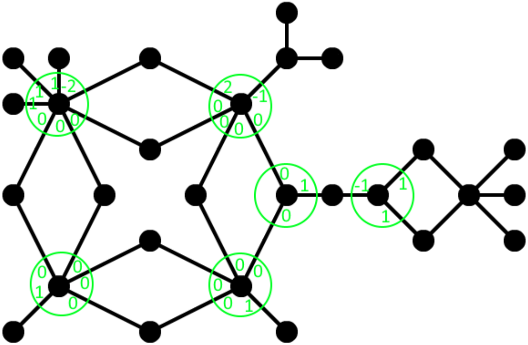

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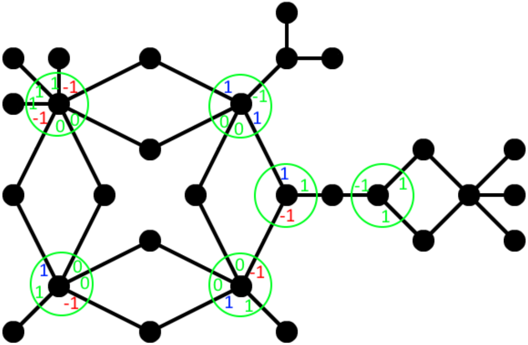

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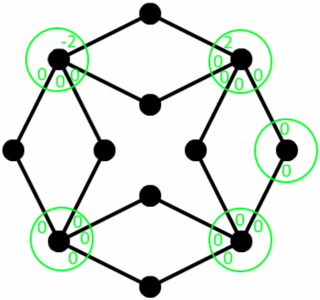

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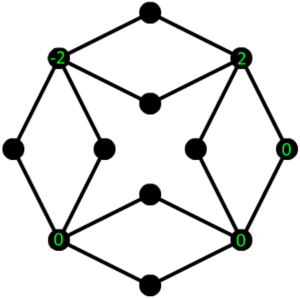

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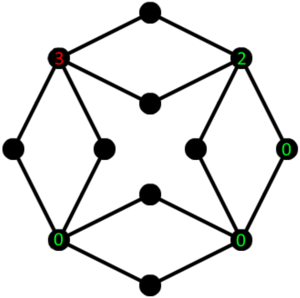

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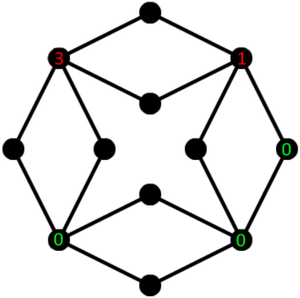

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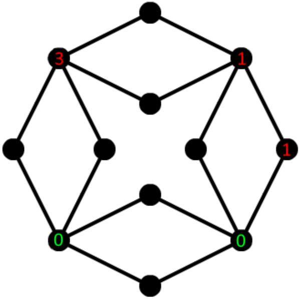

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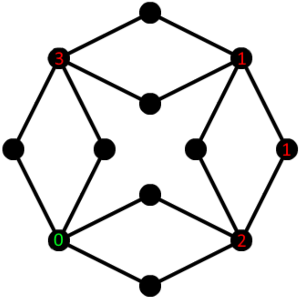

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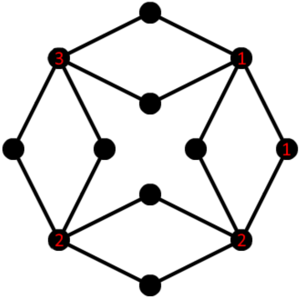

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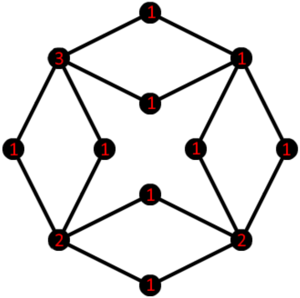

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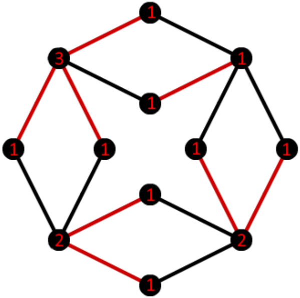

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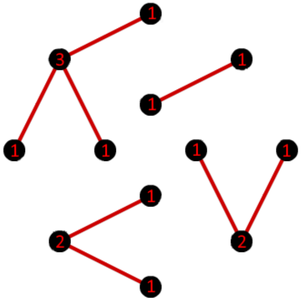

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Theorem

Let G be a bipartite graph with bipartition P_1 and P_2, and let each vertex v of G be assigned an integer value $r(v)$ in the range 0 to $deg(v)$. Then G has a subgraph H with $deg_H(v) = r(v)$ for all vertices v if and only if every $S \subset P_1$ in G satisfies

$$\sum_{v \in S} r(v) \leq \sum_{n \in \Gamma(S)} \min\{r(n), |\Gamma(n) \cap S|\},$$

where $\Gamma(S)$ is the set of neighbours of S in G.

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- *All surplus weights in G are redistributable.*

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- We are currently working on conditions for when a coloured graph G^c is configurable in general.

Richard A. Brualdi, Kathleen P. Kiernan, Seth A. Meyer, Michael W. Schroeder , *Patterns of Alternating Sign Matrices*, Department of Mathematics University of Wisconsin, 2011